ON A QUESTION OF GLASBY, PRAEGER, AND XIA

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ABSTRACT. Recently, Glasby, Praeger, and Xia asked for necessary and sufficient conditions for the 'Jordan Partition' $\lambda(r,s,p)$ to be standard. We give such conditions when p is an odd prime.

1. INTRODUCTION

As usual p is a prime number. There are different ways to explain the notion of Jordan Partition and we approach it via the modular representations of a finite cyclic p-group G of order $q = p^\alpha$ over a field K of characteristic p. It is well-known that there are exactly q isomorphism classes of indecomposable KG-modules. Let $\{V_1, \ldots, V_q\}$ be a set of representatives of these isomorphism classes with $\dim V_i = i$. Many authors have investigated the decomposition of the KG-module $V_m \otimes V_n$, where $m \leq n$, into a direct sum of indecomposable KG-modules — for example, in order of publication, see [6], [11], [8], [9], [10], [7], and [3]. From the works of these authors, it is well-known that $V_m \otimes V_n$ decomposes into a direct sum $V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_m}$ of m indecomposable KG-modules with $\lambda_1 \geq \cdots \geq \lambda_m > 0$, but that the dimensions λ_i of the components depend on the characteristic p. Following [5], we define the **Jordan Partition** $\lambda(m,n,p)$ of mn by

$$\lambda(m,n,p) = (\lambda_1, \ldots, \lambda_m).$$

We say that $\lambda(m,n,p)$ is **standard** iff $\lambda_i = m+n-2i+1$ for $1 \leq i \leq m$.

A sufficient reason for $\lambda(m,n,p)$ to be standard was given in [5, Theorem 2], and Problem 16 of the same paper asked for necessary and sufficient conditions. We give these conditions now when p is odd in the following two theorems which deal with the cases $m < p$ and $m \geq p$, respectively.

Theorem 1. *Assume that p is odd. Define $S = S_1 \cup S_2$, where $S_1 = \{(k,d) \mid 1 \leq k \leq d \leq p+1-k\}$ and*

$$S_2 = \{(k, bp+d) \mid b \geq 1, 1 \leq k \leq (p+1)/2, k-1 \leq d \leq p+1-k\}.$$

If $1 \leq m < p$ and $n \geq m$, then $\lambda(m,n,p)$ is standard iff $(m,n) \in S$.

Theorem 2. *Assume that p is odd. Define*

$$S = \{(ip^t + (p^t \pm 1)/2, jp^t + (p^t \pm 1)/2 + kp^{t+1}) \mid 1 \leq i \leq (p-1)/2, i \leq j \leq p-i-1, k \geq 0\}.$$

Suppose that $p^t \leq m < p^{t+1}$ with $t \geq 1$ and $n \geq m$. Then $\lambda(m,n,p)$ is standard iff $(m,n) \in S$.

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In [3], we gave a recursive definition of the combinatorial object $s_p(m,n)$ and proved that

$$s_p(m,n) = (\lambda_1, \ldots, \lambda_m, \underbrace{0, \ldots, 0}_{n-m}, -\lambda_m, \ldots, -\lambda_1).$$

In Section 2, we will define $s_p(m,n)$, which will be the main tool in our proofs of Theorems 1 and 2 in Section 3.

2. Definition of $s_p(m,n)$

Assume that m and n are positive integers with $m \leq n$. Before we give a formal recursive definition, let us say that $s_p(m,n)$ is a nonincreasing sequence of $m+n$ integers whose first m terms are positive, whose last m terms are negative, and whose middle $n-m$ terms all equal 0. Further, letting $s_p(m,n)(k)$ denote the kth term of $s_p(m,n)$, the sequence is "balanced around its middle" in the sense that

$$s_p(m,n)(m+n+1-k) = -s_p(m,n)(k), \qquad k = 1, \ldots, m+n,$$

and its positive terms sum to $m \cdot n$—so $\sum_{k=1}^{m} s_p(m,n)(k) = m \cdot n$. For example,

$$s_5(6,7) = (12, 10, 8, 5, 5, 2, 0, -2, -5, -5, -8, -10, -12)$$

and

$$s_3(6,8) = (9, 9, 9, 9, 9, 3, 0, 0, -3, -9, -9, -9, -9, -9).$$

The positive terms in $s_p(m,n)$ will turn out to be the dimensions of the indecomposable modules in the decomposition of $V_m \otimes V_n$.

We begin by explaining our notation. All our sequences are finite nonincreasing sequences of integers. If $s = (a_1, \ldots, a_u)$ and $t = (b_1, \ldots, b_v)$ are two sequences with $a_u \geq b_1$, then the sequence $s \oplus t$ is defined by

$$s \oplus t = (a_1, \ldots, a_u, b_1, \ldots, b_v),$$

the concatenation of the two sequences. Following [5], the **negative reverse** $\overline{s}$ of s is defined by $\overline{s} = (-a_u, \ldots, -a_2, -a_1)$. For an integer m and a positive integer k, $(m : k)$ denotes the sequence

$$(\underbrace{m, \ldots, m}_{k}).$$

We will also denote the empty sequence () by $(0 : 0)$. If s is a sequence, then $s_>$ and $s_<$, respectively, denote the subsequences of s consisting of all positive terms, and all negative terms, respectively. For example,

$$s_3(6,8)_< = (-3, -9, -9, -9, -9, -9).$$

For a sequence s and an integer k, $s + k$ denotes the sequence obtained from s by adding k to each of its terms. For example,

$$s_3(6,8)_< + 2 \cdot 3^2 = (15, 9, 9, 9, 9, 9).$$

We now define $s_p(m,n)$ which was introduced in [3].

Definition 1. Let p be a prime and let m and n be integers satisfying $0 \leq m \leq n$. Define $s_p(0,n) = (0 : n)$. Assume now that $0 < m \leq n$ and let k be the unique nonnegative integer such that $p^k \leq n < p^{k+1}$. Write $n = bp^k + d$ with $0 < b < p$ and $0 \leq d < p^k$. Write $m = ap^k + c$ with $0 \leq a < p$ and $0 \leq c < p^k$. Note that $a + c > 0$. We define $s_p(m,n)$ recursively as

$$s_p(m,n) = s_1 \oplus s_2 \oplus s_3,$$

where $s_3 = \overline{s_1}$ and s_1 and s_2 are given in the following exhaustive list of cases.

(1) Case 1: $m+n > p^{k+1}$. Then $s_1 = (p^{k+1} : m+n-p^{k+1})$ and $s_2 = s_p(p^{k+1}-n, p^{k+1}-m)$.
(2) Case 2: $m+n \le p^{k+1}$ and $c+d > p^k$. Then $s_1 = ((a+b+1)p^k : c+d-p^k)$ and $s_2 = s_p((a+b+1)p^k - n, (a+b+1)p^k - m)$.
(3) Case 3: $m+n \le p^{k+1}$, $1 \le c+d \le p^k$, and $a > 0$. Then $s_1 = s_p(\min(c,d), \max(c,d)) + (a+b)p^k$ and $s_2 = s_p((a+b)p^k - n, (a+b)p^k - m)$.
(4) Case 4: $m+n \le p^{k+1}$, $1 \le c+d \le p^k$, $a = 0$ (so $m = c$), and $d > 0$. Then $s_1 = s_p(m, bp^k - d)_< + 2bp^k$ and $s_2 = (0 : n-m)$.
(5) Case 5: $m+n \le p^{k+1}$, $1 \le c+d \le p^k$, $a = 0$, and $d = 0$, so $(m,n) = (c, bp^k)$. Then $s_1 = (bp^k : m)$ and $s_2 = (0 : bp^k - m)$.
(6) Case 6: $m+n \le p^{k+1}$, $c = d = 0$, so $0 < a < a+b \le p$. Then $s_1 = ((a+b-1)p^k : p^k)$ and $s_2 = s_p((a-1)p^k, (b-1)p^k)$.

In Case 6, one can show easily that

$$s_1 = ((a+b-1)p^k : p^k) \oplus ((a+b-3)p^k : p^k) \oplus \cdots \oplus ((b-a+1)p^k : p^k)$$

and $s_2 = (0 : (b-a)p^k)$. When $k = 0$, so $(m,n) = (a,b)$, this specializes to $s_1 = (a+b-1, a+b-3, \ldots, b-a+1)$ and $s_2 = (0 : b-a)$.

Recall that for a sequence s and an integer k, $s(k)$ denotes the kth term of the sequence s. The following result was proved in [3].

Theorem 3. *For positive integers m and n with $m \le n \le q$,*

$$V_m \otimes V_n = \bigoplus_{k=1}^{m} V_{s_p(m,n)(k)}.$$

It follows, as we had stated previously, that

$$\lambda(m,n,p) = (s_p(m,n)(1), s_p(m,n)(2), \ldots, s_p(m,n)(m)).$$

In the next result we characterize exactly when $\lambda(m,n,p)$ is standard for each of the six cases of Definition 1.

Proposition 1. *The Jordan partition $\lambda(m,n,p)$ is standard iff*

(1) *$m+n-p^{k+1} = 1$ and $\lambda(p^{k+1}-n, p^{k+1}-m, p)$ is standard in Case 1*
(2) *$c+d-p^k = 1$ and $\lambda((a+b+1)p^k - n, (a+b+1)p^k - m, p)$ is standard in Case 2*
(3) *$\lambda(\min(c,d), \max(c,d), p)$ is standard, $|c-d| \le 1$ and $\lambda((a+b)p^k - n, (a+b)p^k - m, p)$ is standard in Case 3*
(4) *$\lambda(m, bp^k - d, p)$ is standard in Case 4*
(5) *$m = 1$ in Case 5*
(6) *$k = 0$ in Case 6*

Proof. All except Case 3 are completely obvious. In this case, we note that since $s_p(\min(c,d), \max(c,d))$ and not just $\lambda(\min(c,d), \max(c,d), p)$ is involved in $\lambda(m,n,p)$, if $|c-d| > 1$, then $s_p(\min(c,d), \max(c,d))$ has repeated 0's and so $\lambda(m,n,p)$ has repeated dimensions. □

3. Proofs

First we assemble some lemmas beginning with a special case of Theorem 1.

Lemma 1. *If $1 \leq m, n < p$, then $\lambda(m,n,p)$ is standard iff $m+n \leq p+1$.*

Proof. We can assume that $1 < m \leq n$. In terms of Definition 1, $k = 0$, $c = d = 0$, $m = a$, and $n = b$, leaving us in either Case 1 or Case 6. By the remarks on Case 6 after Definition 1, $\lambda(m,n,p)$ is standard in Case 6. By Proposition 1, $\lambda(m,n,p)$ is standard in Case 1 iff $m+n = p+1$ and $\lambda(p-n, p-n, p)$ is standard. But if $m+n = p+1$, $\lambda(p-n, p-n, p) = \lambda(m-1, n-1, p)$ where $(m-1)+(n-1) = p-1$, that is, Case 6, hence standard. □

Lemma 2. *Here t is a positive integer, $x = (p^t \pm 1)/2$, and $y = (p^t \pm 1)/2$.*

1. *Suppose $x \leq y$. Then for all integers i in the interval $[0, (p-1)/2]$, $\lambda(ip^t + x, ip^t + y, p)$ is standard.*
2. *For any integer b, if $1 \leq b \leq p-1$, then $\lambda(x, bp^t + y, p)$ is standard.*

Proof. (1) By contradiction. Let t be the least positive integer for which this is false. For this t, let i be the least integer for which it is false. Next we show $i > 0$. If $t = 1$, then i cannot be 0 by Lemma 1. If $t > 1$, $x = \frac{p^t \pm 1}{2} = \frac{p-1}{2} p^{t-1} + \frac{p^{t-1} \pm 1}{2}$ and y has a similar expression. Since the result is true for $t-1$, $\lambda(x,y,p)$ is standard. We have shown that $i > 0$.

Only the first three cases of Definition 1 apply but we consider Case 3 first because the other two reduce to this.

Case 3: First $|x-y| \leq 1$. We have just seen that $\lambda(\min(x,y), \max(x,y), p)$ is standard. Since $((i-1)p^t + p^t - \max(x,y), (i-1)p^t + p^t - \min(x,y)) = ((i-1)p^t + x', (i-1)p^t + y')$ where $x' = (p^t \pm 1)/2$ and $y' = (p^t \pm 1)/2$, $\lambda((i-1)p^t + x', (i-1)p^t + y', p)$ is standard by assumption. Hence $\lambda(ip^t + x, ip^t + y, p)$ is standard by Proposition 1.

Case 1: Here $(ip^t + x) + (ip^t + y) > p^{t+1} + 1$. The only possibility is $i = (p-1)/2$ and $x = (p^t+1)/2 = y$. Now $(p^{t+1} - (p-1)/2 \cdot p^t - (p^t+1)/2, p^{t+1} - (p-1)/2 \cdot p^t - (p^t+1)/2) = ((p-1)/2 \cdot p^t + x', (p-1)/2 \cdot p^t + y')$ where $x' = (p^t-1)/2 = y'$. This is a Case 3 situation. Note that $|x'-y'| = 0$ and $\lambda(x', y', p)$ is standard. In addition, $(p-1)p^t - (p-1)/2 \cdot p^t - x' = (p-3)/2 \cdot p^t + p^t - x' = (p-3)/2 \cdot p^t + x$. Hence $\lambda((p-3)/2 \cdot p^t + x, (p-3)/2 \cdot p^t + x, p)$ is standard since $(p-3)/2 < i = (p-1)/2$. By Proposition 1, $\lambda((p-1)/2 \cdot p^t + (p^t+1)/2, (p-1)/2 \cdot p^t + (p^t+1)/2, p)$ is standard.

The treatment of Case 2 is similar to the treatment of Case 1.

(2) By contradiction. Let b the least such integer for which $\lambda(x, bp^t + y, p)$ is not standard. We consider the relevant cases.

Case 4: $x + bp^t + y \leq p^{t+1}$ and $1 \leq x + y \leq p^t$. Then either $x = y = (p-1)/2$ or exactly one of x and y equals $(p-1)/2$ while the other equals $(p+1)/2$. Then since $(x, bp^t - y) = (x, (b-1)p^t + p^t - y)$, $\lambda(x, bp^t - y, p)$ is standard by the definition of b if $b > 1$ or by Part 1 if $b = 1$. It follows that $\lambda(x, bp^t + y, p)$ is standard.

Case 2: $x + bp^t + y \leq p^{t+1}$ but $x + y > p^t$. Here $b \leq p-2$. It must be that $x = y = (p+1)/2$. Since $\lambda(x, bp^t + y, p)$ is not standard, $\lambda((1+b)p^t - bp^t - y, (1+b)p^t - x, p)$, that is, $\lambda(p^t - y, bp^t + p^t - x, p)$ is not standard. This is a Case 4 situation since $(p^t - y) + (p^t - x) < p^t$. Hence $\lambda(p^t - y, bp^t - p^t + x, p) = \lambda(p^t - y, (b-1)p^t + x, p)$ is not standard. This contradicts our choice of b if $b > 1$ and contradicts Part 1 of Lemma 2 if $b = 1$.

Case 1 is handled similarly to Case 2. □

The next two results are just special cases of Proposition 3 of [5] but we will prove them in the setting of Definition 1.

Lemma 3. *If $m < p^t$ and p^t divides n, then $\lambda(m,n,p) = (n,\dots,n)$.*

Proof. Write $n = fp^t$ and proceed by induction on f. When $f = 1$, $m = 0 \cdot p^t + m$ and $p^t = 1 \cdot p^t + 0$, so we are in Case 5, and $\lambda(m,p^t,p) = (p^t,\dots,p^t)$. Hence the result holds when $f = 1$. Now let $f \geq 2$ and assume that the result holds for all integers less than f. Write $fp^t = bp^k + d$ where $0 < b < p$ and $0 \leq d < p^k$. Note that $t \leq k$ and p^t divides d. Here $m = 0 \cdot p^k + m$. We are either in Case 4 if $d > 0$ or Case 5 if $d = 0$. In Case 5,

$$\lambda(m, fp^t, p) = (fp^t,\dots,fp^t) = (n,\dots,.n).$$

In Case 4, $m + d \leq p^k$ and

$$\lambda(m, bp^k + d, p) = s_p(m, bp^k - d)_< + 2bp^k.$$

Since p^t divides $bp^k - d$, $\lambda(m, bp^k - d, p) = (bp^k - d,\dots,bp^k - d)$. Hence $s_p(p^t, bp^k - d)_< = (-bp^k + d,\dots,-bp^k + d)$ and $\lambda(m, bp^k + d, p) = (n,\dots,n)$. □

Lemma 4. *If t and n are positive integers with $p^t \leq n$, then $\lambda(p^t,n,p)$ is not standard.*

Proof. By contradiction. Suppose that n is the least integer $\geq p^t$ for which $\lambda(p^t,n,p)$ is standard. Write $n = bp^k + d$ where $1 < b < p$ and $0 \leq d < p^k$, and write $p^t = ap^k + c$ where $0 \leq a < p$ and $0 \leq d < p^k$. If $t < k$, then $a = 0$ and $c = p^t$; if $t = k$, then $a = 1$ and $c = 0$. We consider the six cases of Definition 1.

Case 1: $p^t + n > p^{k+1}$. If $p^t + n > p^{k+1} + 1$, then $\lambda(p^t,n,p)$ is not standard. We can assume that $p^t + n = p^{k+1} + 1$. In order for $\lambda(p^t,n,p)$ to be standard in this case, $\lambda(p^{k+1} - n, p^{k+1} - p^t, p) = \lambda(p^t - 1, p^{k+1} - p^t, p)$ must be standard. But by Lemma 3, $\lambda(p^t - 1, p^{k+1} - p^t, p) = (p^{k+1} - p^t,\dots,, p^{k+1} - p^t)$ and so is not standard, implying that $\lambda(p^t,n,p)$ is not standard.

Case 2: $p^t + n \leq p^{k+1}$ but $c + d > p^k$. If $c + d > p^k + 1$, $\lambda(p^t,n,p)$ is not standard. We can assume that $c + d = p^k + 1$. In order for $\lambda(p^t,n,p)$ to be standard in this case, $\lambda((a+b+1)p^k - n, (a+b+1)p^k - p^t, p) = \lambda(p^t - 1, (a+b+1)p^k - p^t, p) = \lambda(p^t - 1, n-1, p)$ must be standard. But by Lemma 3, $\lambda(p^t - 1, (a+b+1)p^k - p^t, p) = (n-1,\dots,n-1)$ and so is not standard, implying that $\lambda(p^t,n,p)$ is not standard.

Case 3: $m + n \leq p^{k+1}$, $1 \leq c + d \leq p^k$, and $a > 0$. In this $a = 1$ and $c = 0$. If $|c - d| > 1$, $\lambda(p^t,n,p)$ is not standard. Assume $d = 1$. In order for $\lambda(p^t,n,p)$ to be standard in this case, $\lambda((1+b)p^k - n, (1+b)p^k - m, p) = \lambda(p^k - 1, bp^k, p)$ must be standard. But by Lemma 3, $\lambda(p^k - 1, bp^k, p) = (bp^k,\dots,bp^k)$ and so is not standard, implying that $\lambda(p^t,n,p)$ is not standard.

Case 4: $m + n \leq p^{k+1}$, $1 \leq c + d \leq p^k$, $a = 0$, so $m = c = p^t$, and $d > 0$. In order for $\lambda(p^t,n,p)$ to be standard in this case, $\lambda(p^t, bp^k - d, p)$ must be standard. By since $bp^k - d < n$, this is not standard.

Case 5: $(m,n) = (p^t, bp^k)$ with $t < k$. In this case $\lambda(m,n,p) = (n,\dots,n)$ is not standard.

Case 6: $(m,n) = (p^k, bp^k)$ with $1 + b \leq p$. In this case, $\lambda(m,n,p) = (n,\dots,n)$ is not standard. □

Proof of Theorem 1. First we show that if $(m,n) \in S$, then $\lambda(m,n,p)$ is standard. Notice that by the construction of S, $(m,n) \in S$ implies $m \leq n$.

By contradiction. Let (m,n) be an element of S such that $\lambda(m,n,p)$ is not standard with $m+n$ as small as possible. Then whenever $(m',n') \in S$ with $m'+n' < m+n$, $\lambda(m',n',p)$ is standard. Hence $1 < m < p$ because $\lambda(1,n,p)$ is standard for all n. Since if $(m,n) \in S_1$ then $\lambda(m,n,p)$ is standard by Lemma 1, $m+n > p+1$ and $n = bp+d$ with $b \geq 1$ and $m-1 \leq d \leq p+1-m$. So $d \geq 1$, $m+d \leq p+1$, and $m+p-d \leq p+1$.

Suppose that $p^k \leq n < p^{k+1}$ where $k \geq 1$, and write $n = b_1p^k + d_1$ where $1 \leq b_1 < p$ and $0 \leq d_1 < p^k$. Write $d_1 = rp + d$, where $0 \leq r < p^{k-1}$ and $0 \leq d < p$. So $n = bp + d$ where $b = b_1p^{k-1} + r$. Since $(m,n) \in S$, $m+d \leq p+1$ and $m+p-d \leq p+1$.

We now check out the relevant cases of Definition 1. The fact that $m < p \leq p^k$ rules out Case 3, and the fact that $d > 0$ rules out Cases 5 and 6.

Case 1: $m+n > p^{k+1}$. This implies $m+d > p$, and since $(m,n) \in S$, $m+d = p+1$. Thus $m+n = p^{k+1}+1$, and $b_1 = p-1$. Then

$$(p^{k+1}-n, p^{k+1}-m) = (m-1, n-1) = (m-1, bp+d-1).$$

Note that $m-1+d-1 = p-1$ and $m-1+p-d+1 = m+p-d \leq p+1$. Thus $(m-1, bp+d-1) = (m-1, n-1) \in S$ and since $m-1+n-1 < m+n$, $\lambda(m-1,n-1,p)$ is standard, implying that $\lambda(m,n,p)$ is since by Definition 1, $\lambda(m,n,p)$ consists of the top dimension p^{k+1} and the $m-1$ dimensions of $\lambda(m-1,n-1,p)$.

Case 2: $m+n \leq p^{k+1}$ but $m+d_1 > p^k$. This implies $m+d > p$, and so $m+d = p+1$ and $m+d_1 = p^k+1$. Then

$$((b_1+1)p^k - n, (b_1+1)p^k - m) = (m-1, n-1).$$

As in Case 1, $\lambda(m-1,n-1,p)$ is standard, implying that $\lambda(m,n,p)$ is since by Definition 1, $\lambda(m,n,p)$ consists of the top dimension $(b+1)p^k$ and the $m-1$ dimensions of $\lambda(m-1,n-1,p)$.

Case 4: $m+n \leq p^{k+1}$, $1 \leq m+d_1 \leq p^k$. Since $(m,n) \in S$, we know $m+d \leq p+1$ and $m+p-d \leq p+1$. Also

$$s_p(m, b_1p^k + d_1)_> = s_p(m, b_1p^k - d_1)_< + 2b_1p^k.$$

But $b_1p^k - d_1 = b_1p^k - rp - d = (b_1p^{k-1} - r - 1)p + p - d = (b - 2r - 1)p + p - d$. If $b_1p^k - d_1 = p-d$, then $m \leq p-d$, and so $(m, b_1p^k - d_1) \in S_1 \subset S$. If $b_1p^k - d_1 > p-d$, then $(m, b_1p^k - d_1) \in S_2 \subset S$. Hence, in both cases, $(m, b_1p^k - d_1) \in S$, and so $\lambda(m, b_1p^k - d_1, p)$ is standard, implying $\lambda(m,n,p)$ is as well.

Now we show that if $\lambda(m,n,p)$ is standard with $1 \leq m < p$ and $n \geq m$, then $(m,n) \in S$. We proceed by contradiction. Let $\lambda(m,n,p)$ be standard with $(m,n) \notin S$ such that $m+n$ is as small as possible. Thus whenever $\lambda(m',n',p)$ is standard with $m' < p$ and $m'+n' < m+n$, $(m,n) \in S$. Since $(1,n) \in S$ for every n, $m > 1$. Since an element (k,d) with $1 \leq k \leq d < p$ is standard iff $k+d \leq p+1$, it follows that $n \geq p$, and so $n = bp+d$ with $b \geq 1$ and $0 \leq d < p$. We now check out the relevant cases of Definition 1 and show $m+d \leq p+1$ and $m+p-d \leq p+1$ in each case.

Again suppose that $p^k \leq n < p^{k+1}$ where $k \geq 1$, and write $n = b_1p^k + d_1$ where $1 \leq b_1 < p$ and $0 \leq d_1 < p^k$. Write $d_1 = rp + d$, where $0 \leq r < p^{k-1}$ and $0 \leq d < p$. So $n = bp+d$ where $b = b_1p^{k-1} + r$. Cases 6 and 3 are ruled out because $0 < m < p^k$. Since $m > 1$, we rule out Case 5.

Case 1: $m+n > p^{k+1}$. Then, by Proposition 1, since $\lambda(m,n,p)$ is standard, $m+n = p^{k+1}+1$, implying $m+d = p+1$, and $\lambda(p^{k+1}-n, p^{k+1}-m, p) = \lambda(m-$

$1, n-1, p) = \lambda(m-1, bp+d-1, p)$ is standard, hence in $(p^{k+1}-n, p^{k+1}-m) \in S$. In particular, $m-1+p-(d-1) \leq p+1$, that is, $m+p-d \leq p+1$. We have shown that $(m,n) \in S$.

Case 2: $m+n \leq p^{k+1}$ but $m+d_1 > p^k$. Then, by Proposition 1, since $\lambda(m,n,p)$ is standard, $m+d_1 = p^k+1$, implying $m+d = p+1$, and $\lambda((b_1+1)p^k-n, (b_1+1)p^k-m, p) = \lambda(m-1, n-1, p) = \lambda(m-1, bp+d-1, p)$ is standard, hence in $((b_1+1)p^k-n, (b_1+1)p^k-m) \in S$. Thus $m-1+p-(d-1) \leq p+1$, that is, $m+p-d \leq p+1$. We have shown that $(m,n) \in S$.

Case 4: $m+n \leq p^{k+1}$, $1 \leq m+d_1 \leq p^k$, and $d_1 > 0$. Then $\lambda(m, b_1p^k - d_1, p) = \lambda(m, (b-2r-1)p+p-d, p)$ is standard, hence $(m, b_1p^k-d_1) \in S$. If $(m, b_1p^k-d_1) \in S_2$, then $m+p-d \leq p+1$ and $m+p-(p-d) = m+d \leq p+1$. Hence $(m,n) \in S$. If $(m, b_1p^k-d_1) = (m, p-d) \in S_1$, then $m \leq p-d < m+p-d \leq p+1$, and $(m,n) \in S$ in this case as well.

□

Proof of Theorem 2. First we show that if $(m,n) \in S$ (with $m \leq n$), then $\lambda(m,n,p)$ is standard.

By contradiction. Let (m,n) be an element of S such that $\lambda(m,n,p)$ is not standard with $m+n$ as small as possible. Then whenever $(m',n') \in S$ with $m'+n' < m+n$, $\lambda(m',n',p)$ is standard.

Now we show that if $n < p^{t+1}$ and $(m,n) \in S$, then $\lambda(m,n,p)$ is standard. Since $(m,n) \in S$ and $m \leq n < p^{t+1}$, $(m,n) = (ip^t+x, jp^t+y)$ where $1 \leq i \leq (p-1)/2$, $i \leq j \leq p-i-1$, $x = (p^t \pm 1)/2$, and $y = (p^t \pm 1)/2$ with $x \leq y$ if $j = i$. We go through the cases.

Case 1: $m+n > p^{t+1}$. It must be that $m+n = p^{t+1}+1$ with $i+j = p-1$, and so $x = y = (p^t+1)/2$. Then

$$(p^{t+1}-n, p^{t+1}-m) = (ip^t + (p^t-1)/2, (p-i-1)p^t + (p^t-1)/2) \in S,$$

hence $\lambda(p^{t+1}-n, p^{t+1}-m, p)$ is standard, implying $\lambda(m,n,p)$ is standard since $\lambda(m,n,p)$ consists of the top dimension p^{t+1} followed by the dimensions of $\lambda(p^{t+1}-n, p^{t+1}-m, p)$.

Case 2: $m+n \leq p^{t+1}$ but $x+y > p^t$. This implies $x = y = (p^t+1)/2$. Then

$$((i+j+1)p^t-n, (i+j+1)p^t-m) = (ip^t+(p^t-1)/2, jp^t+(p^t-1)/2) \in S,$$

hence $\lambda((i+j+1)p^t-n, (i+j+1)p^t-m, p)$ is standard, implying $\lambda(m,n,p)$ standard also since $\lambda(m,n,p)$ consists of the top dimension $(i+j+1)p^t$ followed by the dimensions of $\lambda((i+j+1)p^t-n, (i+j+1)p^t-m, p)$.

Case 3: $1 \leq x+y \leq p^t$. There are three cases here. We will treat the case $x = (p^t-1)/2$ and $y = (p^t+1)/2$ — the others are similar. Then

$$((i+j)p^t-n, (i+j)p^t-m) = ((i-1)p^t+(p^t-1)/2, (j-1)p^t+(p^t+1)/2).$$

If $i \geq 2$, then $((i-1)p^t+(p^t-1)/2, (j-1)p^t+(p^t+1)/2) \in S$, hence $\lambda((i-1)p^t + (p^t-1)/2, (j-1)p^t+(p^t+1)/2, p)$ is standard, and so is $\lambda(m,n,p)$. If $i=1$, then $\lambda((p^t-1)/2, (j-1)p^t+(p^t+1)/2, p)$ is standard by Lemma 2, hence $\lambda(m,n,p)$ is.

Cases 4 and 5 do not apply because $i > 0$, and Case 6 does not apply because x and y are positive.

Since we have just shown that $\lambda(m,n,p)$ is standard when $n < p^{t+1}$, we can assume that $n \geq p^{t+1}$.

Suppose that $p^k \leq n < p^{k+1}$ where $k \geq t+1$, and write $n = b_1p^k + d_1$ where $1 \leq b_1 < p$ and $0 \leq d_1 < p^k$. Write $d_1 = rp^t + d$, where $0 \leq r < p^{k-t}$ and

$0 \le d < p^t$. So $n = bp^t + d$ where $b = b_1 p^{k-t} + r$. Since $(m,n) \in S$, $m = ip^t + x$ where $x = (p^t \pm 1)/2$ and $d = (p^t \pm 1)/2$.

Case 1: $m+n > p^{k+1}$. It must be that $x = d = (p^t+1)/2$ and $m+n = p^{k+1}+1$. Then

$$(p^{k+1} - n, p^{k+1} - m) = (m-1, n-1) \in S,$$

hence $\lambda(p^{k+1} - n, p^{k+1} - m, p)$ is standard, implying $\lambda(m,n,p)$ standard.

Case 2: $m + n \le p^{k+1}$, but $m + d_1 > p^k$. Then $x + d > p^t + 1$ and so $x = d = (p^t + 1)/2$. Then

$$((b_1 + 1)p^k - m, (b+1)p^k - m) = (m-1, n-1) \in S,$$

hence $\lambda((b_1+1)p^k - m, (b+1)p^k - m, p)$ is standard, implying $\lambda(m,n,p)$ standard.

Cases 3 and 6 do not apply. Case 5 does not apply since $m \ge p^t \ge p > 1$.

Case 4: Here $m + d_1 \le p^k$ with $d_1 > 0$. We must show that $(m, b_1 p^k - d_1) \in S$. Since $(m,n) \in S$, $n = \ell p^{t+1} + jp^t + y$, where $\ell \ge 0$, $i \le j \le p - i - 1$, and $y = (p^t \pm 1)/2$. We need to compare this representation of n with $n = b_1 p^k + d_1$. Since $\ell p^{t+1} < p^{k+1}$, $\ell < p^{k-t}$, and we can write $\ell = bp^{k-t-1} + e$ where $0 < b < p$ and $0 \le e < p^{k-t-1}$. Therefore $n = bp^k + ep^{t+1} + jp^t + y$. Since

$$ep^{t+1} + jp^t + y \le (p^{k-t-1} - 1)p^{t+1} + jp^t + y = p^k - (p^{t+1} - jp^t - y) < p^k,$$

it follows that $b = b_1$ and $d_1 = ep^{t+1} + jp^t + y$. Therefore

$$\begin{aligned} b_1 p^k - d_1 &= b_1 p^k - ep^{t+1} - jp^t - y \\ &= b_1 p^k - (e-1)p^{t+1} + p^{t+1} - jp^t - y \\ &= (b_1 p^{k-t-1} - e - 1)p^{t+1} + (p - j - 1)p^t + p^t - y. \end{aligned}$$

Since $b_1 p^{k-t-1} - e - 1 \ge 0$ with equality iff $b_1 = 1$ and $e = p^{k-t-1} - 1$, and $i \le p - j - 1 \le p - i - 1$, $(m, b_1 p^k - d_1) \in S$, hence $\lambda(m, b_1 p^k - d_1, p)$ is standard, implying that $\lambda(m,n,p)$ is standard.

Now we show that if (m,n) is standard with $p^t \le m < p^{t+1}$ and $m \le n$, then $(m,n) \in S$.

By contradiction. Let (m,n) be a standard element not in S with $m+n$ as small as possible. Thus every standard (m', n') with $m' \ge p^t$ and $m' + n' < m + n$ is an element of S. Note that $m > p^t$ by Lemma 4.

Now we show that if $n < p^{t+1}$ and (m,n) is standard, then $(m,n) \in S$. Write $m = ap^t + c$ and $n = bp^t + d$ where $1 \le a \le b < p$ and $0 \le c, d < p^t$. We go through the cases.

Case 1: $m + n > p^{t+1}$. By Proposition 1, it must be that $m + n = p^{t+1} + 1$ and that $\lambda(p^{t+1} - n, p^{t+1} - m, p) = \lambda(m-1, n-1, p)$ is standard, hence in S. So $(m-1, n-1) = (ip^t + x, jp^t + y)$ where $x = (p^t \pm 1)/2$, $y = (p^t \pm 1)/2)$. Since $m - 1 + n - 1 = p^{t+1} - 1$, $x + y = p^t - 1$. Hence $x = y = (p^t - 1)/2$, and so $(m,n) = (ip^t + (p^t+1)/2, jp^t + (p^t+1)/2) \in S$.

Case 2: $m + n \le p^{t+1}$ but $c + d > p^t$. By Proposition 1, it must be that $c + d = p^t + 1$ and $\lambda((a+b+1)p^t - n, (a+b+1)p^t - m) = \lambda(m-1, n-1, p)$ is standard, hence in S. So $(m-1, n-1) = (ip^t + x, jp^t + y)$ where $x = (p^t \pm 1)/2$, $y = (p^t \pm 1)/2)$. Since $m-1+n-1 = p^{t+1}-1$, $x+y = p^t-1$. Hence $x = y = (p^t-1)/2$, and so $(m,n) = (ip^t + (p^t+1)/2, jp^t + (p^t+1)/2) \in S$.

Case 3: $1 \le c + d \le p^t$. By Proposition 1, $\lambda(\min(c,d), \max(c,d), p)$ is standard, $|c-d| \le 1$, and $\lambda((a+b)p^t - n, (a+b)p^t - m, p) = \lambda((a-1)p^t + p^t - d, (b-1)p^t + p^t - c, p)$ is standard. If $a \ge 2$, then $((a-1)p^t + p^t - d, (b-1)p^t + p^t - c) \in S$. So

$p^t - d = (p^t \pm 1)/2$ and $p^t - c = (p^t \pm 1)/2$. So either $c = d = (p^t - 1)/2$ or one of c and d is $(p^t-1)/2$ while the other is $(p^t+1)/2$. Thus $(m, n) \in S$. Suppose that $a = 1$. Since $\lambda(p^t - d, (b-1)p^t + p^t - c, p)$ is standard by Lemma 2, $p^t - d + p^t - c \leq p^t + 1$. But $c + d \leq p^t$, so $p^t - d + p^t - c = p^t$ or $p^t - d + p^t - c = p^t + 1$. Now $|c - d| \leq 1$ implies either $p^t - d = p^t - c = (p^t + 1)/2$ or one of $p^t - d$ and $p^t - c$ equals $(p^t - 1)/2$ while the other equals $(p^t + 1)/2$. In any case, $(m, n) \in S$.

Cases 4, 5, and 6 do not apply.

We can assume that $n \geq p^{t+1}$. Suppose that $p^k \leq n < p^{k+1}$ where $k \geq t + 1$, and write $n = b_1 p^k + d_1$ where $1 \leq b_1 < p$ and $0 \leq d_1 < p^k$. Write $d_1 = rp^t + d$, where $0 \leq r < p^{k-t}$ and $0 \leq d < p^t$. So $n = bp^t + d$ where $b = b_1 p^{k-t} + r$. Recall that $m = ip^t + x$ where $x = (p^t \pm 1)/2$.

Case 1: $m+n > p^{k+1}$. It must be than $m+n = p^{k+1}+1$, so $m+d_1 = p^k+1$ and $x+d = p^t+1$. Also $(p^{k+1}-n, p^{k+1}-m) = (m-1, n-1)$ is standard, hence in S. So $(m-1, n-1) = (ip^t + (x-1), jp^t + (d-1) + \ell p^{t+1})$ where $(x-1)+(d-1) = p^t - 1$. Hence $x - 1 = d - 1 = (p^t - 1)/2$ and $(m, n) \in S$.

Case 2: $m + n \leq p^{k+1}$ but $m + d_1 > p^k$. It must be that $m + d_1 = p^k + 1$ and $x + d = p^t + 1$. Also $((b_1 + 1)p^k - n, (b_1 + 1)p^k - m) = (m - 1, n - 1)$ is standard, hence in S. So $(m-1, n-1) = (ip^t + (x-1), jp^t + (d-1) + \ell p^{t+1})$ where $(x - 1) + (d - 1) = p^t - 1$. Hence $x - 1 = d - 1 = (p^t - 1)/2$ and $(m, n) \in S$.

Case 3 does not apply.

Case 4: $1 \leq m+d_1 \leq p^k$ and $d_1 > 0$. Then $(m, b_1 p^k - d_1)$ is standard, hence in S. So $b_1 p^k - d_1 = n - 2d_1 = bp^t + d - 2(rp^t + d) = bp^t - 2rp^t - d = (b_1 p^{k-t} - r - 1)p^t + p^t - d$ where $p^t - d = (p^t \pm 1)/2$. Thus $d = (p^t \pm 1)/2$ and $n = (b_1 p^{k-t} + r)p^t + d$ and $(m, n) \in S$.

Cases 5 and 6 do not apply. □

4. Conclusion

We end with two questions:

(1) What are necessary and sufficient conditions for $\lambda(m, n, 2)$ to be standard?
(2) In [4], we identified generators for the cyclic modules V_{λ_i} in terms of bases for V_m and V_n when $p \geq n + m - 1$. Are these still generators in all the cases when $\lambda(m, n, p)$ is standard?

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