SURFACES OF GENERAL TYPE WITH $K^2 = 2\chi - 1$

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Abstract. We classify minimal algebraic surfaces of general type having $K^2 = 2\chi - 1$ and $\chi \geq 7$. Such surfaces are regular with canonical map of degree one or two. If $p_g \geq 13$ then the surface is a genus two fibration; otherwise we use the canonical map to describe these surfaces as either birational to the canonical image or to a double cover of a rational surface.

1. Introduction

By Noether's inequality minimal surfaces of general type satisfy $K^2 \geq 2\chi - 6$. Horikawa ([7], [8], [9]) classified surfaces with $2\chi - 6 \leq K^2 \leq 2\chi - 4$; surfaces with $K^2 = 2\chi - 3$ have been studied in [18] while the case $K^2 = 2\chi - 2$ is classified in [12].

In this note we consider the case $K^2 = 2\chi - 1$. Murakami [14], [15] has studied such surfaces with non-trivial torsion, in which case $p_g \leq 5$. Here we will assume $p_g \geq 6$ thus our surfaces are torsion-free. By Bombieri ([4] Lemma 14) a surface with $K^2 = 2\chi - 1$ is regular, thus we have $K^2 = 2p_g + 1$.

The main tool in the classification is the canonical map. The degree of the canonical map is either one or two; using these two cases we will show the following classification.

Theorem 1.1. *Let S be a minimal surface of general type over $\mathbb{C}$ such that $K_S^2 = 2\chi - 1$ and $p_g \geq 6$. Then one of the following cases holds.*

(1) *The canonical map of S is birational, $p_g \leq 8$ and the canonical system has at most one isolated base point.*
(2) *S is a genus two fibration and its canonical map factors through an involution with five isolated fixed points.*
(3) *The canonical map of S factors through an involution with three isolated fixed points and $p_g \leq 7$. S is birational to a double cover of a weak del Pezzo surface or a Hirzebruch surface.*
(4) *The canonical map of S factors through an involution with one isolated fixed point and S can be realized as the minimal resolution of a double cover of a Hirzebruch surface; in this case $p_g \leq 12$.*

The paper is organized as follows. In section 2 we show that the canonical map is either birational or of degree two. In the case of a degree two canonical map the image is a rational surface and the canonical involution has 1, 3, or 5 isolated fixed points; an overview of the general properties of the canonical involution is given in section 3. Sections 4 through 6 study the degree two case according to the number of isolated fixed points of the involution.

When the underlying surface is understood, we will write $H^i(D)$ to denote the ith cohomology of the line bundle associated to the divisor D, and $h^i(D)$ for the

2010 Mathematics Subject Classification 14J29 corresponding dimension. The geometric genus is $p_g = h^0(K_S)$ and the irregularity is $q = h^1(\mathcal{O}_S)$; as our surfaces are regular $q = 0$ and the Euler characteristic is $\chi = p_g + 1$.

We write $\equiv$ to denote the linear equivalence of divisors and $|D|$ for the linear system associated to D. We will write Σ_n to denote the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. We call a singularity of a curve *infinitely near* to include the singularity in the proper transform of the curve after blowing up. In particular, an infinitely near triple point is a triple point where all three tangent directions coincide, so that after blowing up the surface at the point, the proper transform of the curve has a triple point on the exceptional divisor.

2. The canonical map

Let S be a minimal surface of general type over $\mathbb{C}$ with $K_S^2 = 2\chi - 1$ and $p_g \geq 6$. As noted above, S is regular thus $K_S^2 = 2p_g + 1$. Write $\varphi : S \to \mathbb{P}^{p_g - 1}$ for the canonical map associated to the system $|K_S|$. By Horikawa ([9] Theorem 1.1) the canonical system $|K_S|$ is not composed with a pencil, thus the image of φ is a surface $\Sigma \subset \mathbb{P}^{p_g-1}$. We can bound the degree of the canonical map φ as follows.

Theorem 2.1. *Let S be a regular surface with $K_S^2 = 2p_g + 1$ and $p_g \geq 6$. Then the degree of the canonical map is at most two.*

Proof. We have

$$K_S^2 = 2p_g + 1 \geq \deg\varphi \, \deg\Sigma \geq \deg\varphi \, (p_g - 2)$$

thus φ must have degree at most three. Moreover, if the degree of φ is equal to three then we have $p_g \leq 7$.

Suppose we are in this case, that is, suppose $\deg\varphi = 3$ and $6 \leq p_g \leq 7$. If $p_g = 6$, Σ is a degree four surface in $\mathbb{P}^5$ and $|K_S|$ has a single basepoint; in the case $p_g = 7$, the system is basepoint free and Σ is a surface of degree five in $\mathbb{P}^6$. However both of these cases contradict Theorem 1.1 of [13]. Thus when $p_g \geq 6$, the canonical map is either birational or of degree two. □

The surfaces with degree two canonical map will be studied in the subsequent sections. In the case where the canonical map is birational we have $K_S^2 \geq 3p_g - 7$ ([8]). Then $K_S^2 = 2p_g + 1$ implies $p_g \leq 8$. Thus we have the three possibilities $p_g = 6, 7, 8$ to consider.

First, when $p_g = 6$ and $K_S^2 = 13$, $K_S^2 = 3p_g - 5$. In this case, $|K_S|$ has no fixed part and at most one basepoint by [12] Lemma 3.5.

If φ is birational and $p_g = 7$, then $K_S^2 = 15$ and $K_S^2 = 3p_g - 6$. In this case Konno [11] has shown that $|K_S|$ is basepoint free.

The case $p_g = 8$ and $K_S^2 = 17$, $K_S^2 = 3p_g - 7$ is described in [1], where the system $|K_S|$ is also shown to be basepoint free. Thus we conclude the first statement of Theorem 1.1: when the canonical map is birational $p_g \leq 8$ and the canonical system has at most one basepoint.

3. The canonical involution

We now turn to the case where the canonical map $\varphi : S \to \Sigma \subset \mathbb{P}^{p_g-1}$ has degree two. Let σ denote the involution induced by φ and let $\pi : S \to S/\sigma$ be the quotient map.

The fixed locus of σ is the union of a smooth, possibly reducible, curve R and k isolated points $P_1, \dots, P_k$. Let $Q_i = \pi(P_i)$ be the image of an isolated fixed point on the quotient surface. The k points Q_i are ordinary double points on S/σ.

Let $V \to S/\sigma$ be the resolution of these double points and write N_i for the -2 curve over Q_i on V. Let $\epsilon : \tilde{S} \to S$ be the blowup of S at the k points P_i. Then σ induces an involution on $\tilde{S}$ with fixed locus equal to the union of R_0, the inverse image of R, and the k exceptional divisors E_i over the P_i. We have the commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\epsilon} & S \\ \tilde{\pi}\downarrow & & \downarrow\pi \\ V & \longrightarrow & S/\sigma \end{array}$$

The map $\tilde{\pi} : \tilde{S} \to V$ is a double cover of V branched along $2L = B + N_1 + \cdots + N_k$, where $\tilde{\pi}^*(B) = R_0$. By standard double cover formulae (see for example [2] and [6]) we obtain the following.

Lemma 3.1. *Using the notation above, let k be the number of isolated fixed points of the involution σ. We have*

(1) $2\left(K_V + L\right)^2 = K_{\tilde{S}}^2 = K_S^2 - k$
(2) $\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_V) + \dfrac{1}{2}\left(L^2 + L \cdot K_V\right)$
(3) $H^i(2K_V + L) = 0$ *for* $i = 1, 2.$
(4) $2K_V + B$ *is nef and big and* $2K_S^2 = \left(2K_V + B\right)^2$

By Beauville [3], the surface V is ruled and therefore rational, since S is regular. The divisor $2K_V + B$ is nef and big follows from $\tilde{\pi}^*\left(2K_V + B\right) = \epsilon^*\left(2K_S\right)$.

Combining the first three statements of the lemma we see

$$\begin{aligned} k &= K_S^2 - 2\left(K_V + L\right)^2 \\ &= K_S^2 + 6\chi(\mathcal{O}_V) - 2\chi(\mathcal{O}_S) - 2h^0(2K_V + L) \\ &= 5 - 2h^0(2K_V + L). \end{aligned}$$

Thus the number of isolated fixed points of the involution σ can be $k = 1, 3$, or 5.

From the lemma we compute

$$B^2 = 4k + 4K_V^2 + 12p_g - 18 \tag{1}$$

and

$$K_V \cdot B = 5 - k - 2p_g - 2K_V^2. \tag{2}$$

By Riemann-Roch and the above we have

$$h^0\left(2K_V + L\right) = \frac{5-k}{2}.$$

Moreover, $h^0\left(3K_V + B\right) = p_g + \dfrac{9-k}{2}$, thus $3K_V + B$ is effective.

As in [6] and [12] we see that by possibly contracting some -1 curves we obtain a surface where the image of $3K_V + B$ is numerically effective.

Lemma 3.2. ([12]) *There is a birational map $f : V \to Y$ from V onto a smooth rational surface Y with canonical divisor K_Y such that B maps to a divisor B_Y on Y with $3K_Y + B_Y$ is nef.*

Proof. If $3K_V + B$ is not nef, then there exists a curve E with $E \cdot (3K_V + B) < 0$ and $E^2 < 0$. Since $2K_V + B$ is nef and big and

$$E \cdot (K_V + 2K_V + B) < 0$$

this implies $E \cdot K_V < 0$, thus E is a -1 curve and $E \cdot B = 2$.

We next show that E does not meet the -2 curves N_i. Since $2L = B + \sum N_i$ and $E \cdot B = 2$, $E \cdot \sum N_i$ is even. For any N_i we have $(E + N_i) \cdot (2K_V + B) = 0$, thus $(E + N_i)^2 = -3 + 2E \cdot N_i < 0$ which implies $E \cdot N_i \leq 1$ for each i.

Thus E meets either two of the nodal curves or none. If E meets two of the nodal curves, say N_1 and N_2, then $(2E+N_1+N_2)^2 = 0$ and $(2E+N_1+N_2)\cdot(2K_V+B) = 0$, a contradiction. Thus $E \cdot N_i = 0$ for each i.

Let $f : V \to Y$ be the contraction of each such curve E. Since $E \cdot B = 2$ the image B_Y of B has a double point at each contracted point. □

Thus the surface V is obtained from Y by blowing up double points of the curve B_Y. As the nodal curves do not meet the exceptional locus, on Y the images of these k nodal curves are still -2 curves. We will continue to write $N_1, \ldots, N_k$ for these curves and we have $B_Y + \sum N_i$ is an even divisor defining the branch locus of a double cover. Also $f^*(2K_Y + B_Y) = 2K_V + B$ thus $2K_Y + B_Y$ is still nef and big. In addition, the formulas (1) and (2) still hold when we replace B and K_V by B_Y and K_Y.

To classify the surfaces S with degree two canonical map, we now consider each of the three cases for k, the number of isolated fixed points of the canonical involution.

4. The case $k = 5$

We first consider the case where the canonical involution σ has five isolated fixed points. Then by Lemma 3.1, $H^0(2K_Y + L) = 0$. This implies that the bicanonical map of S factors through σ and is not birational. In this case S is a genus two fibration ([16], Prop. 3).

Moreover we see that the fibration of genus two curves on S is unique. If $|M_1|$ and $|M_2|$ are distinct genus two pencils, by the index theorem $(M_1 + M_2)^2 K_S^2 \leq ((M_1 + M_2) \cdot K_S)^2$ which reduces to $(M_1 \cdot M_2)^2 K_S^2 \leq 8$, since $M_i \cdot K_S = 2$ and $M_1^2 = M_2^2 = 0$. As $K_S^2 \geq 13$ this implies $M_1 \cdot M_2 = 0$, a contradiction. Thus the fibration on S is unique.

Examples of these surfaces can be constructed as double covers of $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Example 4.1. *Let S be the minimal model of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a curve B of bidegree $(6, 2d)$ for $d \geq 4$. Assume B has five infinitely near triple points and n ordinary order four points. Then S is a surface of general type with $K_S^2 = 2p_g + 1$ where $p_g = 2d - 7 - n$. The pencil of rulings $(0, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to the genus two pencil on S.*

5. The case $k = 3$

We will show when the canonical involution has three isolated fixed points, the surface S can be realized as either a double cover of a del Pezzo or a Hirzebruch surface. These two cases depend on the two possible values of K_Y^2.

Lemma 5.1. *Suppose the involution σ has $k = 3$ isolated fixed points. Then $K_Y^2 = p_g - 4$ or $K_Y^2 = p_g - 3$.*

Proof. When $k = 3$, from Lemma 3.1 we have $K_Y \cdot B_Y = 2 - 2p_g - 2K_Y^2$ and ${B_Y}^2 = 12p_g + 4K_Y^2 - 6$. Since $3K_Y + B_Y$ is nef,

$$\begin{aligned} 0 &\leq (2K_Y + L) \cdot (3K_Y + B_Y) \\ &= 6K_Y^2 + 7K_Y \cdot L + B_Y \cdot L \\ &= 6K_Y^2 + 7\left(1 - p_g - K_Y^2\right) + 6p_g + 2K_Y^2 - 3 \\ &= K_Y^2 - p_g + 4 \end{aligned}$$

thus $K_Y^2 \geq p_g - 4$.

By the index theorem, $K_Y^2 {B_Y}^2 \leq (K_Y \cdot B_Y)^2$ and we have

$$K_Y^2 \left(12p_g + 4K_Y^2 - 6\right) \leq \left(2 - 2p_g - 2K_Y^2\right)^2$$

which reduces to

$$K_Y^2 \leq \frac{(p_g - 1)^2}{p_g + \frac{1}{2}}.$$

This implies $K_Y^2 \leq p_g - 3$. Thus we have two cases, $K_Y^2 = p_g - 4$ or $K_Y^2 = p_g - 3$. □

We now turn to the divisor $4K_Y + B_Y$, which is effective but may not be nef. As in Lemma 3.2, by possibly contracting some curves we can map to a surface where the image of $4K_Y + B_Y$ is numerically effective.

Lemma 5.2. *If $4K_Y + B_Y$ is not nef, then there exists a sequence of blowdowns $\rho : Y \to Z$ such that $4K_Z + B_Z$ is nef.*

Proof. By Riemann Roch and Lemma 3.1 we have $h^0\left(4K_Y + B_Y\right) > 0$ when $k = 3$, thus $4K_Y + B_Y$ is effective. Suppose $4K_Y + B_Y$ is not nef. Then there exists a curve E with $E \cdot (4K_Y + B) < 0$. Since $3K_Y + B_Y$ is nef, we have

$$E \cdot (3K_Y + B_Y) + E \cdot K_Y < 0$$

implies $E \cdot K_Y < 0$ and E must be a -1-curve on Y. This implies $E \cdot B_Y = 3$.

Let N_1, N_2, and N_3 be the -2-curves on Y corresponding to the resolution of the nodes of S/σ. Since $B_Y \equiv 2L - \sum_1^3 N_i$ and $B_Y \cdot E = 3$, we have $E \cdot \sum_1^3 N_i > 0$ and odd.

For each i we have $(E+N_i)\cdot(3K_Y+B_Y) = 0$, so that $(E+N_i)^2 = -3+2E\cdot N_i < 0$ and $E\cdot N_i \leq 1$. As $(E+\sum_1^3 N_i)\cdot(3K_Y+B_Y) = 0$, $(E+\sum_1^3 N_i)^2 = -7+2E\cdot\sum_1^3 N_i < 0$ and $E \cdot \sum_1^3 N_i \leq 3$. Thus E meets either exactly one of the N_i or all three. We now show the latter cannot occur.

Suppose $E \cdot \sum_1^3 N_i = 3$ and consider the divisor $2E + N_1 + N_2$. We have $(2E + N_1 + N_2) \cdot (3K_Y + B_Y) = 0$ and $(2E + N_1 + N_2)^2 = 0$, a contradiction since $3K_Y + B_Y$ is nef. Thus E meets exactly one of the N_i.

When we contract E we obtain a triple point on the image of the branch curve B_Y, since $E \cdot B_Y = 3$. The image of the nodal curve N_i that meets E will be a -1-curve passing through this triple point; contracting this results in an infinitely near triple point on B_Z, the image of B_Y.

□

Next we show in the case $K_Y^2 = p_g - 4$, we will need to contract six such curves E to ensure $4K_Z + B_Z$ is nef. Suppose ρ contracts l curves. We have

$$K_Y \equiv \rho^*(K_Z) + \sum_1^l E_i$$

$$B_Y \equiv \rho^*(B_Z) - 3\sum_1^l E_i$$

thus

$$0 \leq (2K_Z + B_Z) \cdot (4K_Z + B_Z) = 6 - l$$

and $l \leq 6$.

When $K_Y^2 = p_g - 4$, we have $(4K_Y + B)^2 = -6$, thus we must contract at least six curves to obtain a nef divisor. Therefore $l = 6$. We can now classify the surfaces with $K_Y^2 = p_g - 4$.

Theorem 5.3. *Let $K_Y^2 = p_g - 4$. Then $p_g \leq 7$ and S is the minimal resolution of the double cover of a weak del Pezzo surface Z of degree $p_g + 2$ branched over a curve in $|-4K_Z|$ with three infinitely near triple points.*

Proof. Let $\rho : Y \to Z$ be the contraction of 6 -1-curves so that on Z, $4K_Z + B_Z$ is nef. As we saw in Lemma 5.2, the map ρ contracts three curves E_i, each of which meets a corresponding N_i, so that the image of B_Y is the curve B_Z with three infinitely near triple points. We have $K_Z^2 = K_Y^2 + 6 = p_g + 2$ and

$$(2K_Z + B_Z) \cdot (4K_Z + B_Z) = 0.$$

Since $2K_Z + B_Z$ is nef and big and $4K_Z + B_Z$ is effective, we have $2K_Z + \frac{1}{2}B_Z$ is trivial and $-K_Z \equiv K_Z + \frac{1}{2}B_Z$. Thus Z is a weak del Pezzo surface of degree $p_g + 2$ and $p_g \leq 7$. □

For example, we can explicitly construct such surfaces as double covers of the plane.

Example 5.4. *Let B be a degree 12 plane curve with three infinitely near triple points and n ordinary order four points, with $0 \leq n \leq 2$. The minimal resolution of the double cover of $\mathbb{P}^2$ branched along B will have $p_g = 7 - n$ and $K_S^2 = 15 - 2n = 2p_g + 1$. The three -2-curves correspond to the resolution of the three infinitely near triple points. For $n = 1$ and 2, the pencil of lines in $\mathbb{P}^2$ through an order four point of the branch curve corresponds to a genus three pencil on S.*

To complete the classification for $k = 3$ isolated fixed points of the canonical involution of S, we now suppose $K_Y^2 = p_g - 3$. In this case we can show that the system $|4K_Y + B_Y|$ gives a rational pencil.

A computation similar to that for the previous case shows that there is a contraction $\rho : Y \to Z$ of two curves so that the divisor $4K_Z + B_Z$ is nef. By Lemma 5.2 we can write one of these two curves as E while the other is one of the three nodal curves, say N_1, where E is a -1-curve on Y with $B \cdot E = 3$, $E \cdot N_1 = 1$, and $E \cdot N_i = 0$ for $i = 2, 3$. Thus on Z, the image B_Z of the branch curve B has one infinitely near triple point.

By Lemma 3.1, $h^0(4K_Z + B_Z) = 2$, $(4K_Z + B_Z) \cdot K_Z = -2$ and $(4K_Z + B_Z)^2 = 0$, thus the system $|4K_Z + B_Z|$ is a rational pencil. Moreover, $(4K_Z + B_Z) \cdot B_Z = 8$ and we see that S has a hyperelliptic pencil of genus three.

We also have $h^0(2K_Z + L) = 1$; as $N_i \cdot (2K_Z + L) = -1$ for each nodal curve we can write $2K_Z + L = A + N_1 + N_2 + N_3 + E$, where A is a -1-curve with $A \cdot B = 4$, $A \cdot N_1 = A \cdot E = 0$, and $A \cdot N_2 = A \cdot N_3 = 1$.

Let $\rho_1 : Z \to \Sigma_n$ where we contract $8 - K_Z^2 = 9 - p_g$ curves to obtain the Hirzebruch surface Σ_n. Let S_0 represent the pre-image on Z of the $-n$-section of

Σ_n. Then

$$0 \leq (2K_Z + B_Z) \cdot S_0 = (4K_Z + B_Z) \cdot S_0 - 2K_Z \cdot S_0 = 5 - 2n$$

since $K_Z \cdot S_0 = n - 2$, thus $n \leq 2$.

Writing ℓ for the pre-image of the ruling on Σ_n and E_i for each curve contracted by ρ_1, we have

$$\begin{aligned} K_Z &\equiv -2S_0 + (-2-n)\ell + \sum E_i \\ B_Z &\equiv aS_0 + b\ell - \sum n_i E_i \\ 4K_Z + B_Z &\equiv (a-8)S_0 + (b-8-4n)\ell + \sum (4-n_i)E_i \equiv \ell. \end{aligned}$$

Thus $a = 8, b = 9+4n$, and $n_i = 4$ for each i. The branch curve of the double cover can be written as $B_Z \equiv 8S_0 + (9+4n)\ell - \sum 4E_i$; the contracted curves correspond to resolving order four points of the branch curve.

We can choose to contract A, then N_2 to obtain an infinitely near order four point on the image of B_Z. The fiber corresponding to N_3 is then tangent at this point. As there are $8 - K_Z^2 = 9 - p_g$ singularities of order four we have $9 - p_g \geq 2$ thus $p_g \leq 7$.

We have thus shown the following.

Theorem 5.5. *Let $K_Y^2 = p_g - 3$. Then $p_g \leq 7$ and S is the minimal resolution of the double cover of a Hirzebruch surface Σ_n, $n \leq 2$.*

In summary, examples of these surfaces can be constructed as follows.

Example 5.6. *Let $D \equiv 8S_0 + (9+4n)\ell$ on Σ_n with $0 \leq n \leq 2$. We impose one infinitely near triple point and one infintely near order four point on D; moreover we place the order four point so that a fiber ℓ_0 is tangent to D at that point. We also allow D to possibly have k additional order four points. Then resolving these singularities and taking the double cover branched along B, the union of D and ℓ_0, the minimal resolution is a surface S with $p_g = 7 - k$ and $K_S^2 = 15 - 2k = 2p_g + 1$. Note that the pencil $|4K + B|$ corresponds to the ruling of Σ_n; as $\ell \cdot B = 8$ we see that this lifts to a genus three pencil on S.*

6. The case $k = 1$

Lastly we consider the case where the canonical involution has a single isolated fixed point. Let N denote the nodal curve on Y corresponding to the one isolated fixed point of σ; as before we work over Y so we may assume $3K_Y + B_Y$ is nef.

By the index theorem, $K_Y^2 {B_Y}^2 \leq (K_Y \cdot B_Y)^2$ and we obtain $K_Y^2 \leq p_g - 4$. We have

$$0 \leq (2K_Y + L) \cdot (3K_Y + B) = K_Y^2 - p_g + 7$$

thus $K_Y^2 \geq p_g - 7$. By Lemma 3.1, $h^0(4K_Y + B_Y) = 8 + K_Y^2 - p_g$ and $h^0(2K_Y + L) = 2$. Since $(2K_Y + L) \cdot N = -1$, N is a fixed component of the pencil $|2K_Y + L|$ and $h^0(2K_Y + L - N) = 2$ as well. As

$$2(2K + L - N) + N \equiv 4K_Y + B,$$

$h^0(2K_Y + L) \leq h^0(4K_Y + B)$, thus $8 + K_Y^2 - p_g \geq 2$ and $K_Y^2 \geq p_g - 6$. Thus we have $p_g - 6 \leq K_Y^2 \leq p_g - 4$; we will show, in fact, that $K_Y^2 = p_g - 6$ does not occur. To do so, we next consider the moving part $|M|$ of the system $|2K_Y + L|$.

Lemma 6.1. *The moving part $|M|$ of $|2K_Y + L|$ is a rational pencil.*

Proof. The divisor $2K_Y + B_Y$ is big and nef and $(2K_Y + L) \cdot (2K_Y + B_Y) = 5$, thus by the index theorem $M^2 = 0$. We will next show $M \cdot K_Y = -2$.

Since $3K_Y + B_Y$ is nef, we have

$$0 \leq M \cdot (3K_Y + B) \leq (2K_Y + L) \cdot (3K_Y + B) = K_Y^2 - p_g + 7 \leq 3$$

This implies $M \cdot K_Y \leq 1$. To see $M \cdot K_Y < 0$, suppose not. If $K_Y^2 > 0$, then $M \cdot K_Y = 0$ gives a contradiction. As we have $K_Y^2 \geq p_g - 6$, we have $K_Y^2 > 0$ unless $p_g = 6$. However $p_g = 6, K_Y^2 = K_Y \cdot M = 0$ implies $M \cdot B_Y = M \cdot N = 1$, so that M would correspond to a rational pencil on S, a contradiction. Thus we have $K_Y^2 > 0$ and $K_Y \cdot M = -2$. The system $|M|$ is a basepoint free rational pencil on Y. □

We next refine the bound for K_Y^2.

Proposition 6.2. *Suppose the involution σ has one isolated fixed point. Then $K_Y^2 = p_g - 5$ or $K_Y^2 = p_g - 4$.*

Proof. As we have shown above, $p_g - 6 \leq K_Y^2 \leq p_g - 4$. To complete the proof we will show that $K_Y^2 = p_g - 6$ does not occur.

Suppose we have $K_Y^2 = p_g - 6$. By Lemma 3.1, $h^0(4K_Y + B_Y) \geq 8 + K_Y^2 - p_g$. Writing $2(2K + L - N) + N \equiv 4K_Y + B$, we see that $h^0(2M) \leq h^0(4K_Y + B) = 2$. However $|M|$ is a rational pencil, thus $h^0(2M) \geq 3$ and we obtain a contradiction.

Thus we have two cases, $K_Y^2 = p_g - 4$ or $K_Y^2 = p_g - 5$. □

Proposition 6.3. *In the case $K_Y^2 = p_g - 4$, $4K_Y + B_Y$ is nef and $2K_Y + L = M + N$.*

Proof. An argument similar to that following Lemma 5.2 shows that if $K_Y^2 = p_g - 4$, then the effective divisor $4K_Y + B_Y$ is numerically effective. We write $|2K_Y + L| = |M| + N + F$ where M is the moving part of the pencil and F is the (possibly empty) remaining fixed part. We will show $F = 0$ when $K_Y^2 = p_g - 4$.

As $(2K_Y + L) \cdot (4K_Y + B) = 1$, $M \cdot (4K_Y + B) = 1$ and $M \cdot B = 9$. Note that $2(2K_Y + L) - N = 4K_Y + B_Y$, thus $2(M + F) + N = 4K_Y + B_Y$. Since $M^2 = 0$, we have $2M \cdot F + M \cdot N = 1$, thus $M \cdot N = 1$ and $M \cdot F = 0$.

Writing $(M + F)^2 = (2K + L - N)^2 = 0$ we see $F^2 = 0$; thus $M \cdot F = F^2 = 0$ and F is empty.

Therefore $2K_Y + L = M + N$; moreover we have shown that the rational pencil M on Y lifts to a hyperelliptic pencil of genus four on S. □

As Y contains the rational pencil $|M|$, there is a rational map $\rho : Y \to \Sigma_n$ which contracts $8 - K_Y^2 = 12 - p_g$ curves. Thus we have shown the following.

Theorem 6.4. *Suppose $k = 1$ and $K_Y^2 = p_g - 4$. Then $p_g \leq 12$, Y is birational to the Hirzebruch surface Σ_2 and the rational pencil on Y lifts to a genus four pencil on S.*

Moreover we can realize Y by considering the nodal curve N. As $N \cdot M = 1$ the rational map $\rho : Y \to \Sigma_n$ does not contract N. Suppose N meets a -1-curve E. As $M \cdot E = 0$, we compute $E \cdot N = 1$, $E \cdot B_Y = 5$, and there is a reducible fiber $A + E$ of the pencil $|M|$ where A is another -1-curve with $A \cdot E = 1$, $A \cdot B_Y = 4$, and $A \cdot N = 0$. Thus we can choose to contract A which results in an order four point on the branch curve.

We can choose to contract curves that do not meet N. Therefore Y maps to Σ_2 and N maps to the -2-section on the Hirzebruch surface.

Write $B_Y = aS_0 + b\ell - \sum n_i E_i$, where as before ℓ is the pre-image of the ruling on Σ_2 and S_0 represents the -2-section, with $S_0 \equiv N$. The E_i correspond to the exceptional curves contracted by ρ. Using $K_Y = -2S_0 - 4\ell + \sum E_i$ we can write

$$4K_Y + B_Y \equiv (a-8)S_0 + (b-16)\ell + \sum (n_i - 4)E_i \equiv 2M + N,$$

thus $a = 9, b = 18$ and each $n_i = 4$. Thus S can be constructed as the minimal model of the double cover of Σ_2 branched along the union of S_0 and a curve equivalent to $9S_0 + 18\ell$, with $12 - p_g$ order four points.

To complete the classification we turn to the case $K_Y^2 = p_g - 5$.

Proposition 6.5. *In the case $K_Y^2 = p_g - 5$, there is a rational map $\rho : Y \to Z$ contracting a -1-curve E and the image of the nodal curve N so that $4K_Z + B_Z$ is nef and $2K_Y + L = M + N + E$.*

Proof. A similar argument as before shows that contracting two -1-curves results in a nef divisor $4K_Z + B_Z$; moreover, if one of these -1-curves on Y is E, then $E \cdot N = 1$ and contracting E, then N results in the image B_Z of the branch curve B_Y having an infinitely near triple point.

As $N \cdot L = -1$ and $E \cdot (2K_Y + L) = 0$, we can write $2K_Y + L = M + N + E + F$, where F is the remaining fixed part of the system. We will show that F is empty.

As $(2K_Y + L - N - E) \cdot (4K_Y + B) = 0$, $M \cdot (4K_Y + B) = 0$ and $M \cdot B = 8$. As before, $2(2K_Y + L) - N = 4K_Y + B_Y$, thus $2(M + E + F) + N = 4K_Y + B_Y$. Since $M^2 = 0$, we have $2M \cdot E + 2M \cdot F + M \cdot N = 0$, thus $M \cdot N = 0, M \cdot E = 0$, and $M \cdot F = 0$.

Writing $(M+F)^2 = (2K + L - N - E)^2 = 0$ we see $F^2 = 0$; thus $M \cdot F = F^2 = 0$ and F is empty.

Therefore $2K_Y + L = M + E + N$ and the rational pencil $|M|$ corresponds to a hyperelliptic genus three pencil on S. □

Theorem 6.6. *In the case $k = 1$ and $K_Y^2 = p_g - 5$, $p_g \leq 11$ and S is birational to the double cover of a Hirzebruch surface Σ_n, $n \leq 3$.*

Proof. Let $\rho : Y \to \Sigma_n$ be the contraction of E, N, and m additional curves. As we contract $8 - K_Y^2 = 13 - p_g \geq 2$ curves we have $p_g \leq 11$.

As before, let S_0 denote the pre-image of the $-n$ section and ℓ of the ruling on Σ_n. We can write $B_Y = aS_0 + b\ell - 3N - 6E - \sum n_i E_i$ and $K_Y = -2S_0 + (-2 - n)\ell + N + 2E + \sum E_i$. Then

$$4K_Y + B_Y \equiv (a-8)S_0 + (b - 8 - 4n)\ell + N + 2E + \sum (4 - n_i)E_i \equiv 2M + 2E + N,$$

thus $a = 8$, $b = 10 + 4n$, and $n_i = 4$ for each i. The branch curve of the double cover is a member of the system $|8S_0 + (10 + 4n)\ell|$ with one infinitely near triple point and at most m order four points, where $m = 11 - p_g$ The pencil M corresponds to the ruling ℓ; as $\ell \cdot (8S_0 + (10 + 4n)\ell) = 8$ this pencil lifts to a genus three pencil on the double cover.

As in the proof of Theorem 5.5 we can compute

$$0 \leq (2K_Z + B_Z) \cdot S_0 = (4K_Z + B_Z) \cdot S_0 - 2K_Z \cdot S_0 = 6 - 3n$$

since $K_Z \cdot S_0 = n - 2$ and $(4K_Z + B_Z) \cdot S_0 = 2$, thus $n \leq 3$.

□

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