GENERATORS FOR DECOMPOSITIONS OF TENSOR PRODUCTS OF MODULES ASSOCIATED WITH STANDARD JORDAN PARTITIONS

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ABSTRACT. If K is a field of finite characteristic p, G is a cyclic group of order $q = p^\alpha$, U and W are indecomposable KG-modules with $\dim U = m$ and $\dim W = n$, and $\lambda(m,n,p)$ is a standard Jordan partition of mn, we describe how to find a generator for each of the indecomposable components of the KG-module $U \otimes W$.

1. Introduction

Let p be a prime number, K a field of characteristic p, and G a cyclic group of order $q = p^\alpha$, where α is a positive integer. It is well-known that there are exactly q isomorphism classes of indecomposable KG-modules and that such modules are cyclic and uniserial [1, p. 24–25]. Let $\{V_1, \ldots, V_q\}$ be a set of representatives of these isomorphism classes with $\dim V_i = i$. Many authors have investigated the decomposition of the KG-module $V_m \otimes V_n$, where $m \leq n$, into a direct sum of indecomposable KG-modules—for example, in order of publication, see [9], [15], [11], [12], [14], [10], [13], and [3]. From the works of these authors, it is well-known that $V_m \otimes V_n$ decomposes into a direct sum $V_{\lambda_1} \oplus V_{\lambda_2} \cdots \oplus V_{\lambda_m}$ of m indecomposable KG-modules where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$, but that the dimensions λ_i of the components depend on the characteristic p. Now $\lambda(m,n,p) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is called a **Jordan partition** of mn, and $\lambda(m,n,p)$ is said to be **standard** exactly when $\lambda_i = m+n+1-2i$ for every integer $i \in [1,m]$. So when $\lambda(m,n,p)$ is standard,

$$V_m \otimes V_n \cong \bigoplus_{i=1}^{m} V_{n+m+1-2i}.$$

Necessary and sufficient conditions on m, n, and p for $\lambda(m,n,p)$ to be standard were given in [5].

Fix a generator g of G. There is a basis $u_1, u_2, \ldots, u_m$ of V_m on which the action of g is given by $gu_1 = u_1$ and $gu_i = u_{i-1} + u_i$ when $i > 1$. Note that $(g-1)^i u_m = u_{m-i}$, and so u_m generates V_m as a KG-module. Similarly there is a basis $w_1, w_2, \ldots, w_n$ of V_n, with V_n generated as a KG-module by w_n, on which the action of g is given by $gw_1 = w_1$ and $gw_i = w_{i-1} + w_i$ when $i > 1$. Clearly $\{v_{i,j} = u_i \otimes w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis of $V_m \otimes V_n$ over K. Shortly

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we will see that $\mathcal{B} = \{f_{i,j} = u_i \otimes g^{n-i} w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is also a basis of $V_m \otimes V_n$ that turns out to be easier to work with. We will specify, in terms of $\mathcal{B}$, m elements y_1, y_2, ..., y_m in $V_m \otimes V_n$ such that, when $\lambda(m, n, p)$ is standard, $KGy_i \cong V_{n+m+1-2i}$ and $V_m \otimes V_n$ is an internal direct sum of the indecomposable modules KGy_i. $(m \geq 2)$

Barry did this for the special standard partition $\lambda(m, n, p)$ with $m + n \leq p + 1$ in [4], and Glasby, Praeger, and Xia did it for a subset of standard partitions that properly includes the case that Barry dealt with in [6].

We now describe the organization of the paper. In Section 2, we show how to calculate in $V_m \otimes V_n$, state our results in Section 3, work out an example in Section 4, dealt with characteristic 2 in Section 5, and then the rest of the paper deals with the odd characteristic case.

2. Preliminaries

Lemma 1. *The set* $\mathcal{B} = \{f_{i,j} = u_i \otimes g^{n-i} w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ *is an* K*-basis for* $V_m \otimes V_n$*, and* $(g - 1)f_{i,j} = f_{i-1,j} + f_{i,j-1}$*, where we understand that* $f_{k,\ell} = 0$ *if* $k < 1$ *or* $\ell < 1$*.*

Proof. Since $f_{i,j} = v_{i,j} + \sum_{k+\ell<i+j} \alpha_{k,\ell} v_{k,\ell}$, the linear independence of $\mathcal{B}$ follows from the linear independence of $\{v_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Also

$$\begin{aligned} gf_{i,j} &= gu_i \otimes g^{n-i+1} w_j \\ &= (u_i + u_{i-1}) \otimes g^{n-i}(w_j + w_{j-1}) \\ &= u_i \otimes g^{n-i} w_j + u_i \otimes g^{n-i} w_{j-1} + u_{i-1} \otimes g^{n-i}(w_j + w_{j-1}) \\ &= f_{i,j} + f_{i,j-1} + u_{i-1} \otimes g^{n-(i-1)} w_j \\ &= f_{i,j} + f_{i,j-1} + f_{i-1,j} \end{aligned}$$

Thus $(g - 1)f_{i,j} = f_{i,j-1} + f_{i-1,j}$. □

For an integer $k \in [1, m + n - 1]$, define

$$F_k = \langle f_{i,j} \mid i + j \leq k + 1 \rangle, \text{ and } D_k = \langle f_{i,j} \mid i + j = k + 1 \rangle.$$

Also for convenience define $D_k = \{0\}$ for $k < 1$. Note that $F_k = \langle v_{i,j} \mid i+k \leq k+1 \rangle$ and $D_k \subset F_k$.

Lemma 2. *For each integer* $i \in [1, m]$*, define* $x_i = \sum_{j=1}^{i} (-1)^{j-1} f_{j,i+1-j} \in D_i$*. Then* $\{x_1, x_2, \ldots, x_m\}$ *is linearly independent and* $(g - 1)x_i = 0$ *for all* i*.*

Proof. First $\{x_1, x_2, \ldots, x_m\}$ is linearly independent because $x_i \in D_i$ and $F_m = D_1 \oplus \cdots \oplus D_m$. Also

$$\begin{aligned}(g-1)x_i &= \sum_{j=1}^{i}(-1)^{j-1} f_{j,i-j} + \sum_{j=1}^{i}(-1)^{j-1} f_{j-1,i+1-j} \\ &= \sum_{j=1}^{i}(-1)^{j-1} f_{j,i-j} + \sum_{j=0}^{i-1}(-1)^{j} f_{j,i-j} \\ &= f_{i,0} + f_{0,i} \\ &= 0.\end{aligned}$$

□

Then, by Lemma 1, $(g-1)^r(D_k) \subseteq D_{k-r}$ and

$$(g-1)^r(f_{i,j}) = \sum_{k=0}^{r} \binom{r}{k} f_{i+k-r,j-k},$$

where we understand that $f_{i+k-r,j-k} = 0$ if $i+k-r < 0$ or $j-k < 0$.

Hence $(g-1)^{m+n-2k}(D_{m+n-k}) \subseteq D_k$ when $1 \le k \le m$ and

$$(g-1)^{m+n-2k}(f_{i,j}) = \sum_{\ell=0}^{m+n-2k} \binom{m+n-2k}{\ell} f_{i+\ell-m-n+2k,j-\ell}.$$

Assume that $m \le n$. Denote the ordered F-basis

$$(f_{m-k+1,n}, f_{m-k+2,n-1} \ldots, f_{m,n-k+1})$$

of D_{m+n-k} by $\mathcal{B}_{m+n-k}$ and the ordered F-basis $(f_{1,k}, f_{2,k-1}, \ldots, f_{k,1})$ of D_k by $\mathcal{B}_k$. In the case where $m = n$ and $k = m$, $\mathcal{B}_{m+n-k} = \mathcal{B}_k$.

Lemma 3. *Let $A_k(m,n)$ be the matrix with respect to the ordered F-bases $\mathcal{B}_{m+n-k}$ and $\mathcal{B}_k$ of D_{m+n-k} and D_k, respectively. Then*

$$A_k(m,n) = \begin{pmatrix} \binom{m+n-2k}{n-k} & \binom{m+n-2k}{n-k-1} & \cdots & \binom{m+n-2k}{n+1-2k} \\ \binom{m+n-2k}{n+1-k} & \binom{m+n-2k}{n-k} & \cdots & \binom{m+n-2k}{m+2-2k} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m+n-2k}{n-1} & \binom{m+n-2k}{n-2} & \cdots & \binom{m+n-2k}{n-k} \end{pmatrix}.$$

Proof. A typical element in $\mathcal{B}_{m+n-k}$ is $f_{m-k+t,n-t+1}$, $1 \le t \le k$, while a typical element in $\mathcal{B}_k$ is $f_{s,k+1-s}$, $1 \le s \le k$.

Now

$$(g-1)^{m+n-2k}(f_{m-k+t,n-t+1}) = \sum_{\ell=0}^{m+n-2k} \binom{m+n-2k}{\ell} f_{\ell+k+t-n,n-t+1-\ell}$$

When $\ell + k + t - n = s$ (and $n - t - \ell + 1 = k + 1 - s$), $\ell = n + s - k - t$. Thus the coefficient $f_{s,k+1-s}$ in the expansion of $(g-1)^{m+n-2k}(f_{m-k+t,n-t+1})$ is

$$\binom{m+n-2k}{n+s-k-t}.$$

This proves our lemma. □

3. Statement of Results

For an integer $k \in [1, m]$, define the $k \times 1$ column vector C_k to be the coordinate matrix of of x_k with respect to the basis $\mathcal{B}_k$ of D_k. Then C_k consists of alternating 1's and -1's. Then define the $k \times 1$ column vector B_k by $B_k = \mathrm{adj}(A_k)C_k$, where $\mathrm{adj}(A_k(m,n))$ is the classical adjoint of $A_k(m,n)$ (so $A_k(m,n)\mathrm{adj}(A_k(m,n)) = (\det A_k(m,n))I_k = \mathrm{adj}(A_k(m,n))A_k(m,n)$).

Theorem 1. *With $A_k(m,n)$ and B_k defined as above, and y_k defined by*

$$y_k = \sum_{i=1}^{k} b_{i1} f_{n-k+i,m+1-i},$$

the equation $(g-1)^{n+m-2k} \cdot y_k = (\det A_k(m,n))x_k$ holds.

Proof. Since by [2, p. 392],

$$[(g-1)^{n+m-2k} \cdot y_k]_{\mathcal{B}_k} = [(g-1)^{n+m-2k}]_{\mathcal{B}_k, \mathcal{B}_{m+n-k}} [y_k]_{\mathcal{B}_{m+n-k}},$$

we have

$$[(g-1)^{n+m-2k} \cdot y_k]_{\mathcal{B}_k} = A_k(m,n)B_k = A_k(m,n)\mathrm{adj}(A_k(m,n))C_k = (\det A_k(m,n))C_k.$$

But $[x_k]_{\mathcal{B}_k} = C_k$, which implies that $(g-1)^{n+m-2k} \cdot y_k = (\det A_k(m,n))x_k$. □

Corollary 1. *When $\lambda(m,n,p)$ is standard, then $A_k(m,n)$ is invertible for every integer $k \in [1,m]$, and if $y_1, y_2, \ldots, y_m$ are defined as in Theorem 1, $KGy_k \cong V_{n+m+1-2k}$ $(1 \leq k \leq m)$ and*

$$V_m \otimes V_n = KGy_1 \oplus KGy_2 \oplus \cdots \oplus KGy_m.$$

Once we prove that $A_k(m,n)$ is invertible for every integer $k \in [1,m]$, the rest of the proof follows the proof in [4] or [6, Theorem 2].

4. Example

We illustrate Theorem 1 when $m = 4$, $n = 5$, and $k = 3$. In this case $x_3 = f_{1,3} - f_{2,2} + f_{3,1}$,

$$A_3(4,5) = \begin{pmatrix} \binom{3}{2} & \binom{3}{1} & \binom{3}{0} \\ \binom{3}{3} & \binom{3}{2} & \binom{3}{1} \\ \binom{3}{4} & \binom{3}{3} & \binom{3}{2} \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 3 \end{pmatrix},$$

$\det A_3(4,5) = 10$, and

$$\mathrm{adj}(A_3(4,5))C_3 = \begin{pmatrix} 6 & -8 & 6 \\ -3 & 9 & -8 \\ 1 & -3 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ -20 \\ 10 \end{pmatrix}.$$

Thus $y_3 = 20f_{2,5} - 20f_{3,4} + 10f_{4,3}$ and

$$\begin{aligned}
&(g-1)^3 \cdot (20f_{2,5} - 20f_{3,4} + 10f_{4,3}) \\
&\qquad = 20\sum_{k=0}^{3}\binom{3}{k} f_{k-1,5-k} - 20\sum_{k=0}^{3}\binom{3}{k} f_{k,4-k} + 10\sum_{k=0}^{3}\binom{3}{k} f_{k+1,3-k} \\
&\qquad = 20(3f_{1,3} + f_{2,2}) - 20(3f_{1,3} + 3f_{2,2} + f_{3,1}) = 10(f_{1,3} + 3f_{2,2} + 3f_{3,1}) \\
&\qquad = 10f_{1,3} - 10f_{2,2} + 10f_{3,1} \\
&\qquad = (\det A_3(4,5))x_3.
\end{aligned}$$

5. Proof in characteristic 2

By [5], $\lambda(m, n, 2)$ is standard with $1 < m \leq n$ iff either $(m, n) = (2, n)$ with $n \geq 3$ odd or $(m, n) = (3, 6 + 4r)$ where r is a non-negative integer.

When $n \geq 3$ is odd, $A_1(2, n) = (\binom{2+n-2}{n-1} = (n)$ and

$$A_2(2,n) = \begin{pmatrix} \binom{n-2}{n-2} & \binom{n-2}{n-3} \\ \binom{n-2}{n-1} & \binom{n-2}{n-2} \end{pmatrix} = \begin{pmatrix} 1 & n-2 \\ 0 & 1 \end{pmatrix}.$$

Hence both are invertible in K.

When $n = 6+4r$ where r is a non-negative integer, $A_1(3, n) = (\binom{3+n-2}{n-1}) = (\binom{n+1}{n-1})$,

$$A_2(3,n) = \begin{pmatrix} \binom{n-1}{n-2} & \binom{n-1}{n-3} \\ \binom{n-1}{n-1} & \binom{n-1}{n-2} \end{pmatrix} = \begin{pmatrix} n-1 & \binom{n-1}{n-3} \\ 1 & n-1 \end{pmatrix},$$

and

$$A_3(3,n) = \begin{pmatrix} \binom{n-3}{n-3} & \binom{n-3}{n-4} & \binom{n-3}{n-5} \\ \binom{n-3}{n-2} & \binom{n-3}{n-3} & \binom{n-3}{n-4} \\ \binom{n-3}{n-1} & \binom{n-3}{n-2} & \binom{n-3}{n-3} \end{pmatrix} = \begin{pmatrix} 1 & n-3 & \binom{n-3}{n-5} \\ 0 & 1 & n-3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\binom{n+1}{n-1} = 1$ and $\binom{n-1}{n-3} = 0$ in K, these matrices are invertible.

6. Standard Jordan partitions in odd characteristic

For the remainder of this paper, p is a fixed odd prime. Define

$S'_0 = \{(k, d) \in \mathbb{N} \times \mathbb{N} \mid 1 < k \leq d \leq p+1-k\} \cup \{(k, p+k-1) \mid 1 < k \leq (p+1)/2\}$,

and $S_0 = \{(a, b + rp) \mid (a, b) \in S'_0, r \in \{0\} \cup \mathbb{N}\}$.

For an integer $t \geq 1$, define $S'_t = (T_1 \setminus T_2) \cup T_3$ where

$$T_1 = \{(ip^t + (p^t \pm 1)/2, jp^t + (p^t \pm 1)/2) \mid i, j \in \mathbb{N}, 1 \leq i \leq j \leq p - i - 1\},$$

$$T_2 = \{(ip^t + (p^t + 1)/2, ip^t + (p^t - 1)/2) \mid i \in \mathbb{N}, 1 \leq i \leq (p-1)/2\},$$

and

$$T_3 = \{(ip^t + (p^t + 1)/2, ip^t + (p^t - 1)/2 + p^{t+1}) \mid i \in \mathbb{N}, 1 \leq i \leq (p-1)/2\},$$

and $S_t = \{(a, b + rp^{t+1}) \mid (a, b) \in S'_t, r \in \{0\} \cup \mathbb{N}\}$.

Then by [5], $S = \cup_{t\geq 0} S_t$ is the set of ordered pairs (m, n) of positive integers with $1 < m \leq n$ such that $\lambda(m, n, p)$ is standard. Note that for each $(m, n) \in S$, neither m nor n is a power of p.

7. Proof in odd characteristic

To prove Corollary 1, we must show that $A_k(m, n)$, which is a matrix with entries in the prime field of K, is invertible for every integer $k \in [1, m]$ when $(m, n) \in S$.

We know, when $1 \leq k \leq m$, by [6] that

$$d_k(m,n) = \det A_k(m,n) = \prod_{\ell=0}^{k-1} \frac{\binom{m+n-2k+\ell}{n-k}}{\binom{n-k+\ell}{n-k}} = \prod_{\ell=0}^{k-1} \frac{(m+n-2k+\ell)!\ell!}{(n-k+\ell)!(m-k+\ell)!}.$$

In particular, $d_m(m, n) = 1$ because $A_m(m, n)$ is upper triangular with 1's along the diagonal.

Denote the exact power of p dividing a non-zero integer w by $\nu_p(w)$. We will use extensively a theorem of Kummer [8] which states that $\nu_p(\binom{n}{m})$ is the number of 'carries' required to add m and $n - m$ in base-p.

By [7, Lemma 12],

$$\binom{m+n-k-1}{k} d_{k+1}(m,n) = \binom{m+n-2k-2}{n-k-1} d_k(m,n)$$

when $0 \leq k \leq m - 1$, where $d_0(m, n)$ is defined to be 1.

We need to show that $\nu_p(d_k(m, n)) = 0$ for every integer $k \in [1, m]$. Since $d_0(m, n) = 1$, it suffices to prove the following result.

Proposition 1. *For* $(m, n) \in S$*,*

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{m+n-2k-2}{m-k-1}\right) \qquad (*)$$

for every positive integer k *in the interval* $[0, m - 1]$.

Note that since $d_m(m, n) = 1$, we could restrict k to $[0, m-2]$, but in the inductive step, having $k \in [0, m - 1]$ proves useful.

We now outline the plan of the proof. For a non-negative integer t, let $P(t)$ be the statement: Proposition 1 holds for every $(m, n) \in S_t$. First we prove the base case $t = 0$, so $(m, n) \in S_0$, which means that $2 \leq m < p$, in Section 8. Then let $t > 0$ and assume $P(t - 1)$ is true. In Section 9 we use use our inductive hypothesis to show that Proposition 1 holds for $m = n = p^t + \frac{p^t+1}{2}$. In subsequent sections we use this result and our inductive hypothesis to show that Proposition 1 holds for all the other $(m, n) \in S_t$, which are listed below.

(1) $(m, n) = (ip^t + \frac{p^t+1}{2}, ip^t + \frac{p^t+1}{2})$ where $1 < i \leq \frac{p-1}{2}$
(2) $(m, n) = (ip^t + \frac{p^t+1}{2}, jp^t + \frac{p^t+1}{2})$ where $1 \leq i < j \leq p - i - 1$
(3) $(m, n) = (ip^t + \frac{p^t-1}{2}, jp^t + \frac{p^t+1}{2})$ where $1 \leq i \leq j \leq p - i - 1$

(4) $(m,n) = (ip^t + \frac{p^t-1}{2}, jp^t + \frac{p^t-1}{2})$ where $1 \le i \le j \le p-i-1$
(5) $(m,n) = (ip^t + \frac{p^t+1}{2}, jp^t + \frac{p^t-1}{2})$ where $1 \le i < j \le p-i-1$
(6) $(m,n) = (ip^t + \frac{p^t+1}{2}, ip^t + \frac{p^t-1}{2} + p^{t+1})$ where $1 \le i \le \frac{p-1}{2}$
(7) $(m, n + rp^{t+1})$, where (m,n) comes from Cases 1–6 and r is a non-negative integer

Then we will have shown that $P(t-1)$ implies $P(t)$ from which the result follows by appeal to mathematical induction.

8. The case of $m < p$

Proposition 1 holds for $(m,n) \in S_0$.

Proof. Suppose that $1 < m \le n \le p+1-m$. When $k = 0$, $m-1 \le n-1 < m+n-2 \le p-1$. Hence

$$\nu_p\left(\binom{m+n-1}{0}\right) = 0 = \nu_p\left(\binom{m+n-2}{m-1}\right).$$

Now assume $1 \le k \le m-1$. Then

$$m+n-2k-2 < m+n-k-1 \le p-k < p$$

and

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = 0 = \nu_p\left(\binom{m+n-2k-2}{m-k-1}\right)$$

for $k \in [1, m-1]$.

Suppose that $1 < m \le (p+1)/2$, $n = p+m-1$, and $0 \le k \le m-1$. When $k = m-1$, $m+n-k-1 = p+2m-2-(m-1) = p+m-1$, $m+n-2k-1 = p$, $m-k-1 = 0$, and $n-k-1 = p+m-1-m = p-1$. Thus

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{p+m-1}{m-1}\right) = 0 = \nu_p\left(\binom{p-1}{0}\right) = \nu_p\left(\binom{m+n-2k-2}{m-k-1}\right).$$

Assume $0 \le k \le m-2$. Then

$$p+1 \le m+n-2k-2 < m+n-2k-1 \le m+n-k-1 \le p+2m-2 \le 2p-1$$

and

$$p \le n-k-1 \le p+m-2 < 2p.$$

Hence $\nu_p((m+n-2k-1)!) = \nu_p((m+n-2k-2)!) = 1 = \nu_p((m+n-k-1)!) = \nu_p((n-k-1)!)$ in this case.

Clearly $\nu_p(k!) = 0 = \nu_p((m-k-1)!)$. It follows that

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = 0 = \nu_p\left(\binom{m+n-2k-2}{m-k-1}\right)$$

when $0 \le k \le m-2$.

In the remaining part of the proof we use the fact that $\nu_p(\binom{a}{b})$ is the number of carries in adding b and $a-b$ in base p.

Suppose that $1 < m \le n' \le p+1-m$ and that $n = n' + rp$ where r is a positive integer. We must show that then number of carries in adding $m + n - 2k - 1 = m + n' - 2k - 1 + rp$ and k equals the number of carries in adding $m - k - 1$ and $n - k - 1 = n' - k - 1 + rp$. Since, by above, there are none in adding $m + n' - 2k - 1$ and k and none in adding $m - k - 1$ and $n' - k - 1$ the result follows.

Suppose that $1 < m \le (p+1)/2$, $n' = p + m - 1$, and $n = n' + rp$ where r is a positive integer. We must show that then number of carries in adding $m + n - 2k - 1 = m + n' - 2k - 1 + rp$ and k equals the number of carries in adding $m - k - 1$ and $n - k - 1 = n' - k - 1 + rp$. Since, by above, there is no carry in adding $m + n' - 2k - 1$ and k and none in adding $m - k - 1$ and $n' - k - 1$, the result is immediate. □

9. The case of $m = n = p^t + \frac{p^t+1}{2}$

Theorem 2. *Proposition 1 holds for* $m = n = p^t + \frac{p^t+1}{2}$.

We will need two lemmas. First we consider a subcase of $\frac{p^t+1}{2} \le k \le p^t - 1$.

Lemma 4. *When an integer k satisfies* $\frac{p^t+1}{2} \le k \le \frac{p^t+1}{2} + p^{t-1} - 1$,

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{2m-(k-p^{t-1})-1}{k-p^{t-1}}\right) + 1$$

and

$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{2m-2(k-p^{t-1})-2}{m-(k-p^{t-1})-1}\right) + 1.$$

Proof. Since

$$\frac{p^t+1}{2} - 1 = \frac{p^t-1}{2} = \frac{p-1}{2}p^{t-1} + \frac{p-1}{2}p^{t-2} + \cdots + \frac{p-1}{2}p + \frac{p-1}{2},$$

$$\frac{p^t+1}{2} + p^{t-1} - 1 = \frac{p+1}{2}p^{t-1} + \frac{p-1}{2}p^{t-2} + \cdots + \frac{p-1}{2}p + \frac{p-1}{2}.$$

Thus $k = b_{t-1}p^{t-1} + \cdots + b_1 p + b_0$, where $b_{t-1} = \frac{p-1}{2}$ or $b_{t-1} = \frac{p+1}{2}$. Now

$$-\frac{p^t+1}{2} - p^{t-1} + 1 \le -k \le -\frac{p^t+1}{2},$$

implies

$$2p^t + \frac{p^t-1}{2} - p^{t-1} + 1 \le 2m - 1 - k \le 2p^t + \frac{p^t-1}{2}$$

and

$$p^t + (p-2)p^{t-1} + 1 \le 2m - 1 - 2k \le p^t + p^t - 1.$$

Therefore $2m - 2k - 1 = p^t + a_{t-1}p^{t-1} + \cdots + a_1 p + a_0$, where $a_{t-1} = p - 2$ or $a_{t-2} = p - 1$. Note if $k > \frac{p+1}{2}p^{t-1}$, so $b_{t-1} = \frac{p+1}{2}$, then

$$2m - 2k - 1 = 3p^t - 2k < 3p^t - (p^t + p^{t-1}) = p^t + (p-1)p^t,$$

and $a_{t-1} = p - 2$. If $k = \frac{p+1}{2}p^{t-1}$, then $b_{t-1} = \frac{p+1}{2}$ and $a_{t-1} = p - 1$. On the other hand, if $k < \frac{p+1}{2}p^{t-1}$, $b_{t-1} = \frac{p-1}{2}$ and

$$2m - 2k - 1 = 3p^t - 2k > 3p^t - p^t - p^{t-1} = p^t + (p-1)p^{t-1},$$

so $a_{t-1} = p - 1$. For all such k, $a_{t-1} + b_{t-1} \geq p + \frac{p-3}{2} \geq p$. Hence there is a carry from the p^{t-1} digit when we add $2m - 2k - 1$ and k.

Now $2m - 2(k - p^{t-1}) - 1 = 2p^t + a'_{t-1}p^{t-1} + a_{t-2}p^{t-2} + \cdots + a_1 p + a_0$ where $a'_{t-1} = 0$ when $a_{t-1} = p - 2$ and $a'_{t-1} = 1$ when $a_{t-1} = p - 1$. And $k - p^{t-1} = b'_{t-1}p^{t-1} + b_{t-2}p^{t-2} + \cdots + b_1 p + b_0$ where $b'_{t-1} = b_{t-1} - 1$. Hence $a'_{t-1} + b'_{t-1} \leq \frac{p+1}{2}$ with strict inequality unless $k = \frac{p+1}{2}p^{t-1}$. Since $\frac{p+1}{2} < p - 1$ if $p > 3$, there is no carry from the p^{t-1} when we add $2m - 2(k - p^{t-1}) - 1$ and $k - p^{t-1}$ in this case. When $p = 3$ and $k = \frac{p+1}{2}p^{t-1} = 2 \cdot 3^{t-1}$, there is a carry from the 3^{t-1} digit in adding $2m - 2k - 1 = 3^t + 2 \cdot 3^t$ and $k = 2 \cdot 3^{t-1}$ and no carry in adding $2m - 2(k - p^{t-1}) - 1 = 2 \cdot 3^t + 3^{t-1}$ and $k - p^{t-1} = 3^{t-1}$.

From what we have done, it follows that there is exactly one more carry in adding $2m - 2k - 1$ and k than in adding $2m - 2(k - p^{t-1}) - 1$ and $k - p^{t-1}$. This proves

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{2m-(k-p^{t-1})-1}{k-p^{t-1}}\right) + 1$$

when $\frac{p^t+1}{2} \leq k \leq \frac{p^t+1}{2} + p^{t-1} - 1$.

Now $p^t - p^{t-1} \leq m - k - 1 \leq p^t - 1$. Therefore

$$m - k - 1 = (p-1)p^{t-1} + c_{t-2}p^{t-2} + \cdots + c_1 p + c_0,$$

and

$$m - (k - p^{t-1}) - 1 = p^t + 0 \cdot p^{t-1} + c_{t-2}p^{t-2} + \cdots + c_1 p + c_0.$$

Hence there is one more carry in adding $m - k - 1$ to itself than in adding $m - (k - p^{t-1}) - 1$ to itself. This proves

$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{2m-2(k-p^{t-1})-2}{m-(k-p^{t-1})-1}\right) + 1$$

when $\frac{p^t+1}{2} \leq k \leq \frac{p^t+1}{2} + p^{t-1} - 1$. □

Next we handle the rest of the case $\frac{p^t+1}{2} \leq k \leq p^t - 1$.

Lemma 5. *When $\frac{p^t+1}{2} \leq k \leq \frac{p^t+1}{2} + p^{t-1} - 1$ and $k + jp^{t-1} \leq p^t - 1$,*

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{2m-(k+jp^{t-1})-1}{k+jp^{t-1}}\right)$$

and

$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{2m-2(k+jp^{t-1})-2}{m-(k+jp^{t-1})-1}\right).$$

Proof. Using the notation of the previous lemma, $m-k-1 = (p-1)p^{t-1} + c_{t-2}p^{t-2} + \cdots + c_1 p + c_0$. Therefore $m-(k+jp^{t-1})-1 = (p-1-j)p^{t-1} + c_{t-2}p^{t-2} + \cdots + c_1 p + c_0$. Now

$$jp^{t-1} \leq p^t - 1 - k \leq p^t - 1 - \frac{p^t+1}{2} = \frac{p^t-3}{2} = \frac{p-1}{2}p^{t-1} + \frac{p-1}{2}p^{t-2} + \cdots + \frac{p-1}{2}p + \frac{p-3}{2}.$$

Hence $j \leq \frac{p-1}{2}$ and $p - 1 - j \geq \frac{p-1}{2}$. If $j < \frac{p-1}{2}$, then the number of carries in adding $m-k-1$ to itself equals the number of carries in adding $m-(k+jp^{t-1})-1$ to itself.

If $j = \frac{p-1}{2}$, then

$$\frac{p^t+1}{2} \le k \le p^t - 1 - \frac{p-1}{2}p^{t-1} = \frac{p^t + p^{t-1}}{2} - 1$$

and

$$p^t - \frac{p^{t-1}-1}{2} = (p-1)p^{t-1} + \frac{p^{t-1}+1}{2} \le m-k-1 \le p^t - 1.$$

Thus in adding $m-k-1$ to itself there is a carry out of the p^{t-2} digit. Because $p-1-j = \frac{p-1}{2}$, when adding $m-(k+\frac{p-1}{2}p^{t-1})-1$ to itself, that carry out of the p^{t-2} digit causes a carry out of the p^{t-1} digit. We have shown that the number of carries in adding $m-k-1$ to itself equals the number of carries in adding $m-(k+\frac{p-1}{2}p^{t-1})-1$ to itself.

From the previous lemma, $2m-2k-1 = p^t + a_{t-1}p^{t-1} + \cdots + a_1 p + a_0$, where $a_{t-1} = p-2$ or $a_{t-2} = p-1$, and $k = b_{t-1}p^{t-1} + \cdots + b_1 p + b_0$, where $b_{t-1} = \frac{p-1}{2}$ or $b_{t-1} = \frac{p+1}{2}$ and $a_{t-1} + b_{t-1} \ge p + \frac{p-3}{2} \ge p$. Therefore

$$2m - 2(k + jp^{t-1}) - 1 = p^t + (a_{t-1} - 2j)p^{t-1} + a_{t-2}p^{t-2} + \cdots + a_1 p + a_0$$

and $k + jp^{t-1} = (b_{t-1}+j)p^{t-1} + b_{t-2}p^{t-2} + \cdots + b_1 p + b_0$. If $j < \frac{p-1}{2}$, then the number of carries in adding $2m-2k-1$ and k equals the number of carries in adding $2m-2(k+jp^{t-1})-1$ and $k+jp^{t-1}$.

Assume that $j = \frac{p-1}{2}$. Then $\frac{p^t+1}{2} \le k \le p^t - 1 - \frac{p-1}{2}p^{t-1} = \frac{p^t+p^{t-1}}{2} - 1$ from above, clearly $b_{t-1} = \frac{p-1}{2}$, and $a_{t-1} = p-1$ because $p^t + (p-1)p^{t-1} + 2 \le 2m-2k-1 \le 2p^t - 1$.

Then the p^{t-1} digit of $k + \frac{p-1}{2}p^{t-1}$ is $p-1$ and the p^{t-1} digit of $2m-2(k+\frac{p-1}{2}p^{t-1})-1$ is 0. We must show there is a carry out of the p^{t-2} digit in adding $2m-2(k+\frac{p-1}{2}p^{t-1})-1$ and $k+\frac{p-1}{2}p^{t-1}$ which results in a carry out of the p^{t-1} digit. Of course, there must be a carry out of the p^{t-2} digit in adding $2m-2k-1$ and k. It comes down to showing that when $\frac{p^t+1}{2} \le k \le p^t - 1 - \frac{p-1}{2}p^{t-1} = \frac{p^t+p^{t-1}}{2} - 1$,

$$(a_{t-2}p^{t-2} + \cdots + a_1 p + a_0) + (b_{t-2}t^{t-2} + \cdots + b_1 + b_0) \ge p^{t-1}.$$

Now

$$\begin{aligned}
(a_{t-2}p^{t-2} + \cdots + a_1 p + a_0) + (b_{t-2}t^{t-2} + \cdots + b_1 + b_0) \ge p^{t-1} \\
&= (2m-2k-1-p^t-(p-1)p^{t-1}) + (k - \frac{p-1}{2}p^{t-1}) \\
&= 3p^t - k - p^t - (p-1)p^{t-1} - \frac{p-1}{2}p^{t-1} \\
&= \frac{p^t + 3p^{t-1}}{2} - k \\
&\ge \frac{p^t + 3p^{t-1}}{2} - (\frac{p^t + p^{t-1}}{2} - 1) \\
&= p^{t-1} + 1.
\end{aligned}$$

□

Proof of Theorem 2. We break this proof into three cases, the second of which has subcases:

(1) $0 \le k \le \frac{p^t+1}{2} - 1$
(2) $\frac{p^t+1}{2} \le k \le p^t - 1$
(3) $p^t \le k \le m - 1$

Assume that $0 \le k \le \frac{p^t-1}{2}$. When $k = 0$, $\binom{2m-1}{0} = 1$, so $\nu_p(\binom{2m-1}{0}) = 0$. On the other hand $\binom{2m-2}{m-1} = \binom{3p^t-1}{p^t+\frac{p^t-1}{2}}$, and since there are no carries in adding $p^t + \frac{p^t-1}{2}$ to itself, $\nu_p(\binom{2m-2}{m-1}) = 0$. Thus the result of Proposition 1 holds when $k = 0$.

Assume then that $1 \le k \le \frac{p^t-1}{2}$. Then $2m-2k-1 = (3p^t+1)-2k-1 = 2p^t+p^t-2k$,

$$2p^t + 1 \le 2m - 2k - 1 \le 2p^t + (p^t - 2),$$

and

$$2p^t + \frac{p^t+1}{2} \le 2m - k - 1 \le 3p^t - 1.$$

Also, $m - k - 1 = p^t + \frac{p^t+1}{2} - k - 1 = p^t + \frac{p^t-1}{2} - k$ and

$$p^t \le m - k - 1 \le p^t + \frac{p^t-3}{2}.$$

Now the number of carries in adding $2m-2k-1$ and k equals the number of carries in adding $p^t - 2k$ and k, and the number of carries in adding $m - k - 1$ to itself equals the number of carries in adding $\frac{p^t-1}{2} - k$ to itself.

If we set $m' = \frac{p^t+1}{2}$, then $2m' - 2k - 1 = p^t - 2k$ and $m' - k - 1 = \frac{p^t-1}{2} - k$. Then $(m', m') \in S_{t-1}$ and so the result (*) is true for all integers $k \in [1, m' - 1]$ by induction.

Now assume that $\frac{p^t+1}{2} \le k \le p^t - 1$. When $k \le \frac{p^t+1}{2} + p^{t-1} - 1$, $0 \le k - p^{t-1} \le \frac{p^t+1}{2} - 1$. Thus

$$\nu_p\left(\binom{2m-(k-p^{t-1})-1}{k-p^{t-1}}\right) = \nu_p\left(\binom{2m-2(k-p^{t-1})-2}{m-(k-p^{t-1})-1}\right)$$

by our work above. Hence, by Lemma 4,

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{2m-2k-2}{m-k-1}\right)$$

for every integer k satisfying $\frac{p^t+1}{2} \le k \le \frac{p^t+1}{2} + p^{t-1} - 1$.

When $\frac{p^t+1}{2}+p^{t-1} \le k \le p^t-1$, then $k = k'+jp^{t-1}$ where $\frac{p^t+1}{2} \le k' \le \frac{p^t+1}{2}+p^{t-1}-1$ and j is a positive integer. By the previous paragraph,

$$\nu_p\left(\binom{2m-k'-1}{k'}\right) = \nu_p\left(\binom{2m-2k'-2}{m-k'-1}\right).$$

Then, by Lemma 5,

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{2m-2k-2}{m-k-1}\right)$$

for every integer k satisfying $\frac{p^t+1}{2} + p^{t-1} \le k \le p^t - 1$.

Assume now that $p^t \leq k \leq p^t + \frac{p^t+1}{2} - 1$. Then $1 \leq 2m-2k-1 \leq p^t$. When $k = p^t$, $2m-2k-1 = p^t$ and there is no carry in adding $2m-2k-1$ and k and no carry in adding $2m-2(k-p^t)-1$ and $k-p^t = 0$. Assume that $k > p^t$. Then $2m-2k-1 < p^t$ implying $2m-2k-1 = a_{t-1}p^{t-1} + \cdots + a_1 p + a_0$, and $k = p^t + b_{t-1}p^{t-1} + \cdots + b_1 p + b_0$. But $2m-2k-1 = 3p^t - 2k = p^t - 2(k-p^t)$. Thus $p^t - 2(k-p^t) = a_{t-1}p^{t-1} + \cdots + a_1 p + a_0$. Since $k - p^t = b_{t-1}p^{t-1} + \cdots + b_1 p + b_0$, the number of carries in adding $2m-2k-1$ and k equals the number of carries in adding $p^t - 2(k-p^t)$ and $k-p^t$. Thus

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{(p^t+1)-2(k-p^t)-1}{k-p^t}\right)$$

when $p^t \leq k \leq p^t + \frac{p^t+1}{2} - 1$.

On the other hand $m-k-1 = \frac{p^t+1}{2} - (k-p^t) - 1$. Thus

$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{(p^t+1)-2(k-p^t)-2}{\frac{p^t+1}{2}-(k-p^t)-1}\right)$$

when $p^t \leq k \leq p^t + \frac{p^t+1}{2} - 1$. If we set $m' = \frac{p^t+1}{2}$, then $(m', m') \in S_{t-1}$ and hence

$$\nu_p\left(\binom{(p^t+1)-2(k-p^t)-1}{k-p^t}\right) = \nu_p\left(\binom{(p^t+1)-2(k-p^t)-2}{\frac{p^t+1}{2}-(k-p^t)-1}\right).$$

It follows that

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{2m-2k-2}{m-k-1}\right)$$

when $p^t \leq k \leq p^t + \frac{p^t+1}{2} - 1$.

□

10. The case of $m = n = ip^t + \frac{p^t+1}{2}$

The next lemma reduces the case $(m,n) = (ip^t + \frac{p^t+1}{2}, ip^t + \frac{p^t+1}{2})$ to the case $(m,n) = (\frac{p^t+1}{2}, \frac{p^t+1}{2})$ or to the case $(m,n) = (p^t + \frac{p^t+1}{2}, p^t + \frac{p^t+1}{2})$.

Lemma 6. *Here* $m = ip^t + (p^t+1)/2$ *with* $1 \leq i \leq \frac{p-1}{2}$.

(1) *When* $0 \leq j \leq i$ *and* $jp^t \leq k \leq jp^t + \frac{p^t+1}{2} - 1$,

$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{(p^t+1)-2(k-jp^t)-2}{\frac{p^t+1}{2}-(k-jp^t)-1}\right)$$

and

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{(p^t+1)-(k-jp^t)-1}{k-jp^t}\right).$$

(2) *When* $0 \leq j \leq i$ *and* $jp^t + \frac{p^t+1}{2} \leq k \leq (j+1)p^t - 1$,

$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{(3p^t+1)-2(k-jp^t)-2}{p^t+\frac{p^t+1}{2}-(k-jp^t)-1}\right)$$

and

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{(3p^t+1)-(k-jp^t)-1}{k-jp^t}\right).$$

Proof. (1) First suppose that $jp^t \le k \le jp^t + \frac{p^t+1}{2} - 1$. Then

$$(i-j)p^t \le m-k-1 \le (i-j)p^t + \frac{p^t-1}{2},$$

or

$$(i-j)p^t \le ip^t + \frac{p^t-1}{2} - k \le (i-j)p^t + \frac{p^t-1}{2}.$$

Hence

$$0 \le \frac{p^t+1}{2} - 1 - (k-jp^t) \le \frac{p^t+1}{2} - 1.$$

Therefore if $m-k-1 = (i-j)p^t + c_{t-1}p^{t-1} + \cdots + c_1p + c_0$, then

$$ip^t + \frac{p^t+1}{2} - 1 - k = (i-j)p^t + c_{t-1}p^{t-1} + \cdots + c_1p + c_0,$$

and

$$(i-j)p^t + \frac{p^t+1}{2} - 1 - (k-jp^t) = (i-j)p^t + c_{t-1}p^{t-1} + \cdots + c_1p + c_0,$$

from which it follows that

$$\frac{p^t+1}{2} - 1 - (k-jp^t) = c_{t-1}p^{t-1} + \cdots + c_1p + c_0.$$

Hence the number of carries in adding $m-k-1$ to itself equals the number of carries in adding $\frac{p^t+1}{2} - 1 - (k-jp^t$ to itself. Thus

$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{(3p^t+1)-2(k-jp^t)-2}{p^t+\frac{p^t+1}{2}-(k-jp^t)-1}\right).$$

Also

$$2(i-j)p^t + 1 \le 2m-2k-1 \le (2(i-j)+1)p^t,$$

so

$$2ip^t + 1 \le 2m - 2(k-jp^t) - 1 \le (2i+1)p^t,$$

and

$$0 \le k - jp^t \le \frac{p^t+1}{2} - 1.$$

If $k = jp^t$, then $2m-2k-1 = (2(i-j)+1)p^t$, and there are zero carries in adding $2m-2k-1$ and k. Also $2m-2(k-jp^t)-1 = (2i+1)p^t$ and $k-jp^t = 0$, so there are zero carries in adding $2m-2(k-jp^t)-1$ and $k-jp^t$. We can assume that $k > jp^t$, so $2m-2k-1 < (2(i-j)+1)p^t$. Then $2m-2k-1 = 2(i-j)p^t + a_{t-1}p^{t-1} + \ldots a_1p + a_0$ and $k = jp^t + b_{t-1}p^{t-1} + \cdots + b_1p + b_0$. Since $2m-2k-1 = (2i+1)p^t - 2k = 2(i-j)p^t + p^t - 2(k-jp^t)$. Thus $p^t - 2(k-jp^t) = a_{t-1}p^{t-1} + \ldots a_1p + a_0$ and $k - jp^t = b_{t-1}p^{t-1} + \cdots + b_1p + b_0$. This shows that the number of carries in adding $2m-2k-1$ to k equals the number of carries in adding $p^t - 2(k-jp^t)$ to $k-jp^t$. Hence

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{(p^t+1)-(k-jp^t)-1}{k-jp^t}\right).$$

(2) Now suppose that $jp^t + \frac{p^t+1}{2} \le k \le (j+1)p^t - 1$. Note $j < i$ here. Then

$$(2(i-j)-1)p^t + 2 \le 2m - 2k - 1 \le 2(i-j)p^t - 1$$

and $2m - 2k - 1 = (2(i-j)-1)p^t + a_{t-1}p^{t-1} + \cdots + a_1p + a_0$. But

$$2m-2k-1 = (2i+1)p^t - 2k = 2ip^t + p^t - 2k = (2(i-j)-1)p^t + 2p^t - 2(k - jp^t).$$

Therefore $3p^t - 2(k - jp^t) = p^t + a_{t-1}p^{t-1} + \cdots + a_1p + a_0$ and

$$2m - 2(k-jp^t) - 1 = (2i-1)p^t + a_{t-1}p^{t-1} + \cdots + a_1p + a_0.$$

Now $k = jp^t + b_{t-1}p^{t-1} + \cdots + b_1p + b_0$ and $k - jp^t = b_{t-1}p^{t-1} + \cdots + b_1p + b_0$. Hence the number of carries in adding $2m-2k-1$ to k equals the number of carries in adding $3p^t - 2(k-jp^t)$ to $k - jp^t$. Thus

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{(3p^t+1) - (k-jp^t) - 1}{k - jp^t}\right).$$

Now

$$(i-j-1)p^t + \frac{p^t+1}{2} \le m - k - 1 \le (i-j-1)p^t + p^t - 1$$

and

$$m-k-1 = ip^t + \frac{p^t+1}{2} - k - 1 = (i-j-1)p^t + p^t + \frac{p^t+1}{2} - (k-jp^t) - 1.$$

If $m - k - 1 = (i-j-1)p^t + c_{t-1}p^{t-1} + \cdots + c_1p + c_0$, then

$$p^t + \frac{p^t+1}{2} - (k-jp^t) - 1 = c_{t-1}p^{t-1} + \cdots + c_1p + c_0.$$

This shows that the number of carries in adding $m-k-1$ to itself equals the number of carries in adding $p^t + \frac{p^t+1}{2} - (k-jp^t) - 1$ to itself. Hence

$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{(3p^t+1) - 2(k-jp^t) - 2}{p^t + \frac{p^t+1}{2} - (k-jp^t) - 1}\right).$$

□

Corollary 2. *Proposition 1 holds for* $(m,n) = (ip^t + \frac{p^t+1}{2}, ip^t + \frac{p^t+1}{2})$.

Proof. When $jp^t \le k \le jp^t + \frac{p^t+1}{2} - 1$,

$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{(p^t+1) - 2(k-jp^t) - 2}{\frac{p^t+1}{2} - (k-jp^t) - 1}\right)$$

and

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{(p^t+1) - (k-jp^t) - 1}{k-jp^t}\right).$$

Set $m' = \frac{p^t+1}{2}$. Then $(m',m') \in S_{t-1}$, so Proposition 1 holds for (m',m') by inductive assumption and

$$\nu_p\left(\binom{(p^t+1) - 2(k-jp^t) - 2}{\frac{p^t+1}{2} - (k-jp^t) - 1}\right) = \nu_p\left(\binom{(p^t+1) - (k-jp^t) - 1}{k-jp^t}\right).$$

Thus
$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{2m-k-1}{k}\right)$$
for such k.

When $jp^t + \frac{p^t+1}{2} \leq k \leq (j+1)p^t - 1$,
$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{(3p^t+1)-2(k-jp^t)-2}{p^t+\frac{p^t+1}{2}-(k-jp^t)-1}\right)$$
and
$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{(3p^t+1)-(k-jp^t)-1}{k-jp^t}\right).$$
Set $m' = p^t + \frac{p^t+1}{2}$. Then Proposition 1 holds for (m', m') by Theorem 2. Hence
$$\nu_p\left(\binom{(3p^t+1)-2(k-jp^t)-2}{p^t+\frac{p^t+1}{2}-(k-jp^t)-1}\right) = \nu_p\left(\binom{(3p^t+1)-(k-jp^t)-1}{k-jp^t}\right).$$
Thus
$$\nu_p\left(\binom{2m-2k-2}{m-k-1}\right) = \nu_p\left(\binom{2m-k-1}{k}\right)$$
for all such k. □

11. The case of $(m,n) = (ip^t + \frac{p^t+1}{2}, jp^t + \frac{p^t+1}{2})$

Here $1 \leq i \leq j \leq p-1-i$, so $i \leq \frac{p-1}{2}$.

Lemma 7. *For every integer $k \in [0, m-1]$,*
$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{2m-k-1}{k}\right)$$
and
$$\nu_p\left(\binom{m+n-2k-2}{m-k-1}\right) = \nu_p\left(\binom{2m-2k-2}{m-k-1}\right).$$

Proof. If $2m-2k-1 = a_tp^t + a_{t-1}p^{t-1} + \dots a_1p + a_0$ and $k = b_tp^t + b_{t-1}p^{t-1} + \dots + b_1p + b_0$, then $m+n-2k-1 = (a_t+j-i)p^t + a_{t-1}p^{t-1} + \dots a_1p + a_0$. If $k=0$, then the number of carries in adding $m+n-1$ and 0 equals the number of carries in adding $2m-1$ and 0. Assume that $k>0$, in which case $a_t \leq 2i \leq i+j \leq p-1$ and $m+n-k-1 < p^{t+1}$. Thus the number of carries in adding $m+n-2k-1$ and k equals the number of carries in adding $2m-2k-1$ and k, and therefore
$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{2m-k-1}{k}\right).$$
If $m-k-1 = c_tp^t + c_{t-1}p^{t-1} + \dots + c_1p + c_0$, then $n-k-1 = (c_t+j-i)p^t + c_{t-1}p^{t-1} + \dots + c_1p + c_0$ and $m+n-2k-2 \leq m+n-2 < p^{t+1}$. Thus the number of carries in adding $m-k-1$ and $n-k-1$ equals the number of carries in adding $m-k-1$ to itself, and therefore
$$\nu_p\left(\binom{m+n-2k-2}{m-k-1}\right) = \nu_p\left(\binom{2m-2k-2}{m-k-1}\right).$$

□

Corollary 3. *Proposition 1 holds for* $(m,n) = (ip^t + \frac{p^t+1}{2}, jp^t + \frac{p^t+1}{2})$.

Proof. Since, by Corollary 2,

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{2m-2k-2}{m-k-1}\right)$$

for every integer $k \in [0, m-1]$, the result follows from Lemma 7. □

12. Interlude

In the next three sections, $m = ip^t + \frac{p^t \pm 1}{2}$, $n = jp^t + \frac{p^t \pm 1}{2}$, $m \le n$, and $(m,n) \in S'_t$. We will relate $\nu_p(\binom{m+n-k-1}{k})$ to $\nu_p(\binom{m_1+n_1-k-1}{k})$ and $\nu_p(\binom{m+n-2k-2}{m-k-1})$ to $\nu_p(\binom{m_1+n_1-2k-2}{m_1-k-1})$ where $(m_1, n_1) = (ip^t + \frac{p^t+1}{2}, jp^t + \frac{p^t+1}{2})$. Since by Corollary 3, the result holds for (m_1, n_1), it will follow immediately for (m,n).

13. The case of $(m,n) = (ip^t + \frac{p^t-1}{2}, jp^t + \frac{p^t+1}{2})$

Here $1 \le i \le j \le p-1-i$.

Lemma 8. *Let* $m_1 = m+1$. *For every integer* $k \in [0, m-1]$,

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{m_1+n-k-1}{k}\right)$$

and

$$\nu_p\left(\binom{m+n-2k-2}{m-k-1}\right) = \nu_p\left(\binom{m_1+n-2k-2}{m_1-k-1}\right).$$

Proof. Here $m_1 + n = (i+j+1)p^t + 1$. Then $m+n-2k-1 = m_1+n-2(k+1) = (i+j+1)p^t - 2k - 1$. We must show that the number of carries in adding $m_1+n-2(k+1)$ and k equals the number of carries in adding $m_1+n-2k-1$ and k. It suffices to show that

$$\nu_p\left(\binom{(i+j+1)p^t-k-1}{k}\right) = \nu_p\left(\binom{(i+j+1)p^t-k}{k}\right)$$

for every integer $k \in [0, m-1]$. Note that

$$\binom{(i+j+1)p^t-k}{k} = \frac{(i+j+1)p^t-k}{(i+j+1)p^t-2k}\binom{(i+j+1)p^t-k-1}{k}.$$

Since p is an odd prime, the exact power of p that divides $(i+j+1)p^t - k$ equals the exact power of p that divides $(i+j+1)p^t - 2k$ and the result follows.

Also, $m-k-1 = m_1-k-2$ and $m+n-2k-2 = m_1+n-2k-3 = (i+j+1)p^t-2k-2$. We must show

$$\nu_p\left(\binom{(i+j+1)p^t-2k-2}{m_1-k-2}\right) = \nu_p\left(\binom{(i+j+1)p^t-2k-1}{m_1-k-1}\right).$$

for every integer $k \in [0, m_1 - 2]$. Note that

$$\binom{(i+j+1)p^t - 2k - 1}{m_1 - k - 1} = \frac{(i+j+1)p^t - 2k - 1}{m_1 - k - 1} \binom{(i+j+1)p^t - 2k - 2}{m_1 - k - 2}.$$

Since $(i+j+1)p^t + 1 - 2k - 2 = (j-i)p^t + (2i+1)p^t + 1 - 2k - 2 = (j-i)p^t + 2(m_1 - k - 1)$ and p is an odd prime, the exact power of p dividing $(i+j+1)p^t - 2k - 1$ equals the exact power of p dividing $m_1 - k - 1$ and the result follows. □

14. The case of $(m, n) = (ip^t + \frac{p^t+1}{2}, jp^t + \frac{p^t-1}{2})$

Here $1 \leq i < j \leq p - 1 - i$.

Lemma 9. *Let $n_1 = n + 1$. For every integer $k \in [0, m-1]$,*

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{m+n_1-k-1}{k}\right)$$

and

$$\nu_p\left(\binom{m+n-2k-2}{m-k-1}\right) = \nu_p\left(\binom{m+n_1-2k-2}{m-k-1}\right).$$

Proof. The first equality above follows from the first equality in Lemma 8. Here $m + n = (i + j)p^t$. Since

$$\binom{m+n_1-2k-2}{m-k-1} = \frac{m+n_1-2k-2}{n_1-k-1} \binom{m+n-2k-2}{m-k-1}$$

and

$$\frac{m+n_1-2k-2}{n_1-k-1} = \frac{(i-j)p^t + 2(n_1-k-1)}{n_1-k-1},$$

$\nu_p((i-j)p^t + 2(n_1 - k - 1)) = \nu_p(n_1 - k - 1)$ and the second equality holds. □

15. The case of $(m, n) = (ip^t + \frac{p^t-1}{2}, jp^t + \frac{p^t-1}{2})$

Here $1 \leq i \leq j \leq p - 1 - i$.

Lemma 10. *Let $m_1 = m + 1$, $n_1 = n + 1$. For every integer $k \in [0, m-1]$,*

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{m_1+n_1-(k+1)-1}{k+1}\right)$$

and

$$\nu_p\left(\binom{m+n-2k-2}{m-k-1}\right) = \nu_p\left(\binom{m_1+n_1-2(k+1)-2}{m_1-(k+1)-1}\right).$$

Proof. Now $m + n = m_1 + n_1 - 2 = (i + j + 1)p^t - 1$. We must show that

$$\nu_p\left(\binom{(i+j+1)p^t-k-2}{k}\right) = \nu_p\left(\binom{(i+j+1)p^t-k-1)}{k+1}\right)$$

for every integer $k \in [0, m-1]$. Note that

$$\binom{(i+j+1)p^t-k-1)}{k+1} = \frac{(i+j+1)p^t-k-1}{k+1} \binom{(i+j+1)p^t-k-2}{k}.$$

Since the exact power of p dividing $(i+j+1)p^t - k - 1$ equals the exact power of p dividing $k+1$, the result follows. Since $m+n-2k-2 = m_1+n_1-2(k+1)-2$ and $m-k-1 = m_1-(k+1)-1$, the second result is clear. □

16. The case of $(m,n) = (ip^t + \frac{p^t+1}{2}, ip^t + \frac{p^t-1}{2} + p^{t+1})$

Lemma 11. *For every integer $k \in [0, m-1]$,*

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{2m-k-1}{k}\right)$$

and

$$\nu_p\left(\binom{m+n-2k-2}{m-k-1}\right) = \nu_p\left(\binom{2m-2k-2}{m-k-1}\right).$$

Proof. Let $n_1 = n+1$. First we show that for every integer $k \in [0, m-1]$,

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{m+n_1-k-1}{k}\right)$$

and

$$\nu_p\left(\binom{m+n-2k-2}{m-k-1}\right) = \nu_p\left(\binom{m+n_1-2k-2}{m-k-1}\right).$$

Here $m+n = 2ip^t + p^{t+1}$. Now

$$\binom{m+n_1-k-1}{k} = \frac{m+n_1-k-1}{m+n_1-2k-1}\binom{m+n-k-1}{k}$$

and

$$\frac{m+n_1-k-1}{m+n_1-2k-1} = \frac{2ip^t+p^{t+1}-k}{2ip^t+p^{t+1}-2k}.$$

Clearly $\nu_p(2ip^t+p^{t+1}-k) = \nu_p(2ip^t+p^{t+1}-2k)$ and the first equality holds.

Now

$$\binom{m+n_1-2k-2}{m-k-1} = \frac{m+n_1-2k-2}{n_1-k-1}\binom{m+n-2k-2}{m-k-1}$$

and

$$\frac{m+n_1-2k-2}{n_1-k-1} = \frac{-p^{t+1}+2(n_1-k-1)}{n_1-k-1}.$$

Clearly $\nu_p(-p^{t+1}+2(n_1-k-1)) = \nu_p(n_1-k-1)$ and the second equality follows.

Finally, since $m+n_1 = 2m+p^{t+1}$ and there are no carries into the p^{t+1} digit in adding $m+n_1-2k-1$ and k or in adding n_1-k-1 and $m-k-1$,

$$\nu_p\left(\binom{m+n_1-k-1}{k}\right) = \nu_p\left(\binom{2m-k-1}{k}\right)$$

and

$$\nu_p\left(\binom{m+n_1-2k-2}{m-k-1}\right) = \nu_p\left(\binom{2m-2k-2}{m-k-1}\right),$$

the result follows.

□

Corollary 4. *Proposition 1 holds for $(m,n) = (ip^t + \frac{p^t+1}{2}, ip^t + \frac{p^t-1}{2} + p^{t+1})$.*

Proof. Since, by Corollary 2,

$$\nu_p\left(\binom{2m-k-1}{k}\right) = \nu_p\left(\binom{2m-2k-2}{m-k-1}\right)$$

for every integer $k \in [0, m-1]$, the result follows from Lemma 11. □

17. The case of $(m, n' + rp^{t+1})$ where $(m, n') \in S'_t$

Let $m = ip^t + \frac{p^t \pm 1}{2}$ and $n' = jp^t + \frac{p^t \pm 1}{2}$ with $m \leq n'$ $(m-1 \leq n'?)$ and $i+j \leq p-1$.

Note that $m + n' - k - 1 \leq p^{t+1}$ with equality only occurring when $k = 0$, $m = ip^t + \frac{p^t+1}{2}$, $n' = jp^t + \frac{p^t+1}{2}$, and $i + j = p - 1$. Thus in adding $m - k - 1$ and $n' - k - 1$, there are no carry into the p^{t+1} digit. Hence the number of carries in adding $m - k - 1$ and $n - k - 1$ equals the number of carries in adding $m - k - 1$ and $n' - k - 1$. Thus

$$\nu_p\left(\binom{m+n-2k-2}{m-k-1}\right) = \nu_p\left(\binom{m+n'-2k-2}{m-k-1}\right)$$

for every integer $k \in [0, m-1]$. Clearly, $\binom{m+n-0-1}{0} = 1 = \binom{m+n'-0-1}{0}$. Assume that $1 \leq k \leq m-1$. Then there is no carry into the p^{t+1} digit in adding $m + n' - 2k - 1$ and k. Hence the number of carries in adding $m + n - 2k - 1$ and k equals the number of carries in adding $m + n' - 2k - 1$ and k. Thus

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{m+n'-k-1}{k}\right)$$

for every integer $k \in [0, m-1]$. Since

$$\nu_p\left(\binom{m+n'-k-1}{k}\right) = \nu_p\left(\binom{m+n'-2k-2}{m-k-1}\right),$$

it follows that

$$\nu_p\left(\binom{m+n-k-1}{k}\right) = \nu_p\left(\binom{m+n-2k-2}{m-k-1}\right)$$

for every integer $k \in [0, m-1]$.

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