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Generators for decompositions of tensor products of modules

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Abstract. If K is a field of finite characteristic p, G is a cyclic group of order $q = p^{\alpha}$, U and W are indecomposable KG-modules, and $p \geq \dim U + \dim W - 1$, we describe how to find a generator for each of the indecomposable components of the KG-module $U \otimes W$.

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1. Introduction. Let p be a prime number, K a field of characteristic p, and G a cyclic group of order $q = p^{\alpha}$, where α is a positive integer. It is well-known that there are exactly q isomorphism classes of indecomposable KG-modules and that such modules are cyclic and uniserial [1, p. 24–25]. Let $\{V_1, \ldots, V_q\}$ be a set of representatives of these isomorphism classes with $\dim V_i = i$. Many authors have investigated the decomposition of the KG-module $V_n \otimes V_m$, where $n \geq m$, into a direct sum of indecomposable KG-modules—for example, in order of publication, see [3–10]. From the works of these authors, it is well-known that $V_n \otimes V_m$ decomposes into a direct sum of m indecomposable KG-modules, but that the dimensions of the components depend on the characteristic p. However, in the special case of $p \geq n + m - 1$ that we consider in this paper,

$$V_n \otimes V_m \cong \bigoplus_{i=1}^{m} V_{n+m+1-2i}.$$

Fix a generator g of G. There is a basis $u_1, u_2, \ldots, u_n$ of V_n on which the action of g is given by $gu_1 = u_1$ and $gu_i = u_{i-1} + u_i$ when $i > 1$. Note that $(g-1)^i u_n = u_{n-i}$, and so u_n generates V_n as a KG-module. Similarly there is a basis $w_1, w_2, \ldots, w_m$ of V_m, with V_m generated as a KG-module by w_m, on which the action of g is given by $gw_1 = w_1$ and $gw_i = w_{i-1} + w_i$ when $i > 1$.

Clearly $\mathcal{B} = \{v_{i,j} = u_i \otimes w_j \mid 1 \le i \le n, 1 \le j \le m\}$ is a basis of $V_n \otimes V_m$. We will specify, in terms of $\mathcal{B}$, m elements $y_1, y_2, \ldots, y_m$ in $V_n \otimes V_m$ such that, when $p \ge n+m-1$, $KGy_i \cong V_{n+m+1-2i}$ and $V_n \otimes V_m$ is an internal direct sum of the indecomposable modules KGy_i.

Our approach is based on a result in [2]. When $1 \le k \le m$, define

$$x_k = \sum_{i=1}^{k} \sum_{j=1}^{k+1-i} (-1)^{k-j} \binom{k-j}{i-1} v_{i,j},$$

alternatively

$$x_k = \sum_{\ell=2}^{k+1} \sum_{i=1}^{\ell-1} (-1)^{k+i-\ell} \binom{k+i-\ell}{i-1} v_{i,\ell-i}.$$

In [2], the present author showed that, independent of p, $\{x_k \mid 1 \le k \le m\}$ is a basis over K for the fixed space of G's action on $V_n \otimes V_m$.

In this paper, we will define elements $y_1, y_2, \ldots, y_m$ in $V_n \otimes V_m$ and elements $\alpha_1, \alpha_2, \ldots, \alpha_m$ in the prime field of K such that $(g-1)^{n+m-2i}(y_i) = \alpha_i x_i$, and so $(g-1)^{n+m-2i+1}(y_i) = 0$ when $1 \le i \le m$. Each y_i will be written as a linear combination of the $v_{k,\ell}$ with coefficients in the prime field of K that are sums of signed products of binomial coefficients; similarly, each α_i will be defined as a sum of signed products of binomial coefficients. In particular, the definitions of the y_i and α_i will not depend on p. When $p \ge n+m-1$, we will show that $\alpha_i \ne 0$, implying that $(g-1)^{n+m-2i}(y_i) \ne 0$. In this case—$p \ge n+m-1$—then it will follow that

$$V_n \otimes V_m = KGy_1 \oplus KGy_2 \oplus \cdots \oplus KGy_m,$$

where KGy_i is an indecomposable KG-module of dimension $n+m+1-2i$.

When $p \ge n+m-1$, if $z_i = \alpha_i^{-1} y_i$, then $(g-1)^{n+m-2i}(z_i) = x_i$, but we prefer our formulation in the previous paragraph because it is "over $\mathbb{Z}$."

The paper is organized as follows: in the following section we state our results, in Section 3 we work out an example, and in Section 4 we prove our results.

2. Statement of main results. It will be helpful to think of elements of the basis $\mathcal{B} = \{v_{i,j} = u_i \otimes w_j \mid 1 \le i \le n, 1 \le j \le m\}$ of $V_n \otimes V_m$ as forming the rectangular array below:

$$\begin{array}{cccc} v_{n,1} & v_{n,2} & \cdots & v_{n,m} \\ v_{n-1,1} & v_{n-1,2} & \cdots & v_{n-1,m} \\ \vdots & \vdots & \ddots & \vdots \\ v_{2,1} & v_{2,2} & \cdots & v_{2,m} \\ v_{1,1} & v_{1,2} & \cdots & v_{1,m} \end{array}$$

Define the *weight* of a basis element $v_{i,j}$ to be $i+j$. For $\ell = 1, \ldots, n+m-1$, denote by X_ℓ the subspace of $V_n \otimes V_m$ spanned by basis elements of weight at most $\ell+1$.

In this rectangular array the set $\mathcal{B}_k$ of basis elements involved in x_k form the right triangle below:

$$\begin{array}{ccccc} v_{k,1} & & & & \\ v_{k-1,1} & v_{k-1,2} & & & \\ \vdots & \vdots & \ddots & & \\ v_{2,1} & v_{2,2} & \cdots & v_{2,k-1} & \\ v_{1,1} & v_{1,2} & \cdots & v_{1,k-1} & v_{1,k} \end{array}$$

Clearly X_k is the subspace spanned by these elements and X_k has a filtration

$$\{0\} \subset X_1 \subset X_2 \subset \cdots \subset X_k.$$

We will identify y_k as a linear combination of the basis elements in the right triangle below:

$$\begin{array}{ccccc} v_{n,m-k+1} & & & & \\ v_{n-1,m-k+1} & v_{n-1,m-k+2} & & & \\ \vdots & \vdots & \ddots & & \\ v_{n-k+2,m-k+1} & v_{n-k+2,m-k+2} & \cdots & v_{n-k+2,m-1} & \\ v_{n-k+1,m-k+1} & v_{n-k+1,m-k+2} & \cdots & v_{n-k+1,m-1} & v_{n-k+1,m} \end{array}$$

Denote by X'_k the subspace spanned by the set $\mathcal{B}'_k$ of these elements. Then X'_k has a filtration

$$\{0\} \subset X'_1 \subset X'_2 \subset \cdots \subset X'_k$$

where $X'_\ell = X'_k \cap X_{n+m-2k+\ell}$.

We order the bases $\mathcal{B}$, $\mathcal{B}_k$, and $\mathcal{B}'_k$ as follows: $v_{i,j} < v_{k,l}$ if and only $i+j < k+l$ or if $i+j = k+l$ then $i < k$. For example, the ordering of $\mathcal{B}'_k$ is

$$\begin{array}{l} v_{n-k+1,m-k+1}, \\ v_{n-k+1,m-k+2}, v_{n-k+2,m-k+1}, \\ \cdots, \\ v_{n-k+1,m}, v_{n-k+2,m-1}, \ldots, v_{n,m-k+1}. \end{array}$$

When $1 \leq k \leq m$, define the $\frac{k(k+1)}{2} \times \frac{k(k+1)}{2}$ matrix A_k by

$$A_k = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \cdots & A_{1,k} \\ O & A_{2,2} & A_{2,3} & \cdots & A_{2,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_{k,k} \end{pmatrix}$$

where $A_{t,r}$ with $t \le r$ is the $t \times r$ matrix given by

$$A_{t,r} = \binom{n+m-2k}{n+m-2k+t-r} \times \begin{pmatrix} \binom{n+m-2k+t-r}{m-k} & \binom{n+m-2k+t-r}{m-k-1} & \cdots & \binom{n+m-2k+t-r}{m-k-r+1} \\ \binom{n+m-2k+t-r}{m-k+1} & \binom{n+m-2k+t-r}{m-k} & \cdots & \binom{n+m-2k+t-r}{m-k-r+2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+m-2k+t-r}{m-k+t-1} & \binom{n+m-2k+t-r}{m-k+t-2} & \cdots & \binom{n+m-2k+t-r}{m-k+t-r} \end{pmatrix}.$$

Note when $n = m$, then A_m is the identity matrix.

For k in the integer range $[1, m]$, define the $\frac{k(k+1)}{2} \times 1$ column vector C_k to be the coordinate matrix of x_k with respect to the basis $\mathcal{B}_k$ of X_k, that is, $C_k = [x_k]_{\mathcal{B}_k}$. Thus the single entry in row $(\ell-1)(\ell-2)/2 + i$ of C_k, where $2 \le \ell \le k+1$ and $1 \le i \le \ell - 1$, is $(-1)^{k+i-\ell}\binom{k+i-\ell}{i-1}$, that is, $c_{(\ell-1)(\ell-2)/2+i,1}$ is the coefficient of $v_{i,\ell-i}$ in the expression of x_k. Finally, define the $\frac{k(k+1)}{2} \times 1$ column vector D_k by $D_k = \operatorname{adj}(A_k)C_k$, where $\operatorname{adj}(A_k)$ is the classical adjoint of A_k.

Theorem 1. *With A_k, C_k, and D_k defined as above, and y_k defined by*

$$y_k = \sum_{\ell=n+m-2k+2}^{n+m-k+1} \sum_{i=n-k+1}^{\ell-(m-k+1)} d_{(\ell-n-m+2k-1)(\ell-n-m+2k-2)/2+i-n+k,1} v_{i,\ell-i},$$

where $d_{(\ell-n-m+2k-1)(\ell-n-m+2k-2)/2+i-n+k,1}$ is the single entry in row $\frac{(\ell-n-m+2k-1)(\ell-n-m+2k-2)}{2} + i - n + k$ of D_k, the equation

$$(g-1)^{n+m-2k} \cdot y_k = (\det A_k)x_k$$

holds.

Corollary 1. *If $p \ge n+m-1$, then A_k is invertible, and if $y_1, y_2, \ldots, y_m$ are defined as Theorem 1,*

$$V_n \otimes V_m = KGy_1 \oplus KGy_2 \oplus \cdots \oplus KGy_m.$$

3. Example. In this section we will consider the case when $n = 5$ and $m = 3$. Before we do so, we record a lemma that allows us to calculate the action of $(g-1)^r$ on any basis element $v_{a,b}$. Its proof is straightforward and based on

$$(g-1) \cdot v_{a,b} = v_{a-1,b-1} + v_{a-1,b} + v_{a,b-1},$$

with $v_{i,j} = 0$ if $i \le 0$ or $j \le 0$. Essentially the same result is stated in [10, Lemma 1].

Lemma 1. *If a, b, and r are positive integers, then*

$$(g-1)^r \cdot v_{a,b} = \sum_{\ell=\max(z,2)}^{a+b-r} \binom{r}{\ell-z} \times \sum_{j=r-a+1}^{\ell+r-a-1} \binom{\ell-z}{j} v_{a+j-r,\ell+r-a-j}, \quad \text{where } z = a+b-2r.$$

The variable ℓ here indexes weight.

In this case $x_1 = v_{1,1}$, $x_2 = -v_{1,1} + v_{1,2} - v_{2,1}$, and $x_3 = v_{1,1} - v_{1,2} + 2v_{2,1} + v_{1,3} - v_{2,2} + v_{3,1}$.

Also $A_1 = (\binom{6}{6}\binom{6}{2})$,

$$A_2 = \begin{pmatrix} \binom{4}{4}\binom{4}{1} & \binom{4}{3}\binom{3}{1} & \binom{4}{3}\binom{3}{0} \\ 0 & \binom{4}{4}\binom{4}{1} & \binom{4}{4}\binom{4}{0} \\ 0 & \binom{4}{4}\binom{4}{2} & \binom{4}{4}\binom{4}{1} \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} \binom{2}{2}\binom{2}{0} & \binom{2}{1}\binom{1}{0} & 0 & \binom{2}{0}\binom{0}{0} & 0 & 0 \\ 0 & \binom{2}{2}\binom{2}{0} & 0 & \binom{2}{1}\binom{1}{0} & 0 & 0 \\ 0 & \binom{2}{2}\binom{2}{1} & \binom{2}{2}\binom{2}{0} & \binom{2}{1}\binom{1}{1} & \binom{2}{1}\binom{1}{0} & 0 \\ 0 & 0 & 0 & \binom{2}{2}\binom{2}{0} & 0 & 0 \\ 0 & 0 & 0 & \binom{2}{2}\binom{2}{1} & \binom{2}{2}\binom{2}{0} & 0 \\ 0 & 0 & 0 & \binom{2}{2}\binom{2}{2} & \binom{2}{2}\binom{2}{1} & \binom{2}{2}\binom{2}{0} \end{pmatrix}.$$

Hence $\det A_1 = 15$, $\det A_2 = 40$, and $\det A_3 = 1$. Now $C_1 = (1)$ and $D_1 = \operatorname{adj}(A_1)C_1 = (1)(1) = (1)$. Thus $y_1 = v_{5,3}$, and $(g-1)^6 \cdot y_1 = 15v_{1,1} = (\det A_1)x_1$ by Lemma 1. Next

$$C_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \operatorname{adj}(A_2)C_2 = \begin{pmatrix} -30 \\ 20 \\ -40 \end{pmatrix}.$$

Thus $y_2 = -30v_{4,2} + 20v_{4,3} - 40v_{5,2}$ and

$$\begin{aligned} (g-1)^4 \cdot y_2 &= (g-1)^4(-30v_{4,2} + 20v_{4,3} - 40v_{5,2}) \\ &= -30(4v_{1,1}) + 20(12v_{1,1} + 4v_{1,2} + 6v_{2,1}) - 40(4v_{1,1} + v_{1,2} + 4v_{2,1}) \\ &= -40v_{1,1} + 40v_{1,2} - 40v_{2,1} \\ &= (\det A_2)x_2, \end{aligned}$$

using Lemma 1. Finally

$$C_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \operatorname{adj}(A_3)C_3 = \begin{pmatrix} 6 \\ -3 \\ 12 \\ 1 \\ -3 \\ 6 \end{pmatrix}.$$

Thus $y_3 = 6v_{3,1} - 3v_{3,2} + 12v_{4,1} + v_{3,3} - 3v_{4,2} + 6v_{5,1}$ and, again by Lemma 1,

$$\begin{aligned}(g-1)^2 \cdot y_3 &= (g-1)^2(6v_{3,1} - 3v_{3,2} + 12v_{4,1} + v_{3,3} - 3v_{4,2} + 6v_{5,1}) \\ &= 6v_{1,1} - 3(2v_{1,1} + v_{1,2} + 2v_{2,1}) + 12v_{2,1} \\ &\quad + (v_{1,1} + 2v_{1,2} + 2v_{2,1} + v_{1,3} + 2v_{2,2} + v_{3,1}) \\ &\quad - 3(2v_{2,1} + v_{2,2} + 2v_{3,1}) + 6v_{3,1} \\ &= v_{1,1} - v_{1,2} + 2v_{2,1} + v_{1,3} - v_{2,2} + v_{3,1} \\ &= (\det A_3)x_3.\end{aligned}$$

Assume $p \geq 7$. Then $\dim KGy_k = 9-2k$ and $\mathrm{soc}KGy_k = Kx_k$ for $1 \leq k \leq 3$. Since the socles are linearly independent over K, $KGy_1 + KGy_2 + KGy_3$ is a direct sum and because the dimensions sum to 15,

$$V_5 \otimes V_3 = KGy_1 \oplus KGy_2 \oplus KGy_3.$$

4. Proofs of Theorem 1 and Corollary 1.

Proof of Theorem 1. We must show when $1 \leq k \leq m$ that

1. $(g-1)^{n+m-2k}(X'_k) \subset X_k$
2. the matrix of $(g-1)^{n+m-2k}_{|X'_k}$ with respect to the ordered bases $\mathcal{B}'_k$ and $\mathcal{B}_k$ of X'_k and X_k is A_k, that is, $[(g-1)^{n+m-2k}_{|X'_k}]_{\mathcal{B}_k,\mathcal{B}'_k} = A_k$
3. $(g-1)^{n+m-2k} \cdot y_k = (\det A_k)x_k$

Since weights of elements in $\mathcal{B}'_k$ range from $n+m-k+1$ down to $n+m-2k+2$, and $n+m-k+1-(n+m-2k) = k+1$, it follows by Lemma 1 that $(g-1)^{n+m-2k}(X'_k) \subset X_k$.

Next we calculate the $\frac{k(k+1)}{2} \times \frac{k(k+1)}{2}$ matrix A of $(g-1)^{n+m-2k}_{|X'_k}$ with respect to the ordered bases $\mathcal{B}'_k$ of X'_k and $\mathcal{B}_k$ of X_k.

In $\mathcal{B}'_k$, weights of elements range from $n+m-2k+2$ to $n+m-k+1$. For $r \in [1,k]$, the elements of weight $n+m-2k+1+r$ are $v_{n-k+s,m-k+1+r-s}$ where s ranges from 1 to r. We must calculate

$$(g-1)^{n+m-2k} \cdot v_{n-k+s,m-k+1+r-s}.$$

By Lemma 1, this equals

$$\sum_{\ell=2}^{r+1} \binom{z}{\ell+z-r-1} \sum_{j=m-k-s+1}^{\ell+m-k-s-1} \binom{\ell+z-r-1}{j} v_{j-m+k+s,\ell+m-k-s-j},$$

where $z = n+m-2k$. The coefficients here appear in the column $\frac{r(r-1)}{2}+s$ of A. Note that only basis elements of weight at most $r+1$ appear in this formula. Since the basis elements of weight at most $r+1$ occupy the first $\frac{r(r+1)}{2}$ positions in the ordered basis $\mathcal{B}_k$, this implies that $a_{i,\frac{r(r-1)}{2}+s} = 0$ if $i > \frac{r(r+1)}{2}$. Assume that $i \leq \frac{r(r+1)}{2}$. In $\mathcal{B}_k$, the elements of weight $t+1$, where $1 \leq t \leq r$, are $v_{b,t+1-b}$ where $1 \leq b \leq t$. We will calculate the coefficient of $v_{b,t+1-b}$ in the expansion of $(g-1)^{n+m-2k} \cdot v_{n-k+s,m-k+1+r-s}$.

This involves solving the system $j-m+k+s=b$ and $\ell+m-k-s-j=t+1-b$ for ℓ and j. The answer is $\ell=t+1$ and $j=m-k-s+b$, yielding the coefficient

$$\binom{n+m-2k}{n+m-2k+t-r}\binom{n+m-2k+t-r}{m-k-s+b}.$$

This entry occurs in row $\frac{t(t-1)}{2}+b$ of column $\frac{r(r-1)}{2}+s$ of A. Thus

$$a_{\frac{t(t-1)}{2}+b,\frac{r(r-1)}{2}+s}=\binom{n+m-2k}{n+m-2k+t-r}\binom{n+m-2k+t-r}{m-k-s+b}.$$

This proves that the matrix of $(g-1)^{n+m-2k}$ with respect to the ordered bases $\mathcal{B}'_k$ and $\mathcal{B}_k$ of V'_k and V_k is A_k.

Since

$$[(g-1)^{n+m-2k}(y_k)]_{\mathcal{B}_k}=[(g-1)^{n+m-2k}]_{\mathcal{B}_k,\mathcal{B}'_k}[y_k]_{\mathcal{B}'_k},$$

we have

$$[(g-1)^{n+m-2k}(y_k)]_{\mathcal{B}_k}=A_kD_k=A_k\mathrm{adj}(A_k)C_k=(\det A_k)C_k.$$

But $[x_k]_{\mathcal{B}_k}=C_k$, and this implies that $(g-1)^{n+m-2k}\cdot y_k=(\det A_k)x_k$. □

We need the following lemma for our proof of Corollary 1. We feel that this result must be known but, as we could not find it in the literature, we offer a proof.

Lemma 2. *If n, m, and k are positive integers with $k\le m\le n$ and $A(n,m,k)$ is the $k\times k$ matrix with ijth entry $\binom{n}{m+i-j}$, then*

$$\det A(n,m,k)=\prod_{i=0}^{k-1}\binom{n+i}{m+i}\prod_{i=1}^{k-1}\frac{i!}{(n-m+i)^{k-i}}.$$

Proof. By induction on k. The result is clearly true when $k=1$. Let $k\ge 2$ and assume the result is true for $k-1$. Now

$$A(n,m,k)=\begin{pmatrix}\binom{n}{m} & \binom{n}{m-1} & \cdots & \binom{n}{m-k+1}\\ \binom{n}{m+1} & \binom{n}{m} & \cdots & \binom{n}{m-k+2}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{n}{m+k-2} & \binom{n}{m+k-3} & \cdots & \binom{m}{m-1}\\ \binom{n}{m+k-1} & \binom{n}{m+k-2} & \cdots & \binom{n}{m}\end{pmatrix}.$$

Subtracting $\frac{n-m-k+2}{m+k-1}$ times row $k-1$ from row k results in a new kth row of

$$\left(0,\binom{n+1}{m+k-2}\frac{1}{m+k-1},\binom{n+1}{m+k-3}\frac{2}{m+k-1},\ldots,\binom{n+1}{m}\frac{k-1}{m+k-1}\right).$$

Subtracting $\frac{n-m-k+3}{m+k-2}$ times row $k-2$ from row $k-1$ results in a new $(k-1)$th row of

$$\left(0, \binom{n+1}{m+k-3}\frac{1}{m+k-2}, \binom{n+1}{m+k-4}\frac{2}{m+k-2}, \dots, \binom{n+1}{m-1}\frac{k-1}{m+k-2}\right)$$

Continuing in this way we end by subtracting $\frac{n-m}{m+1}$ times row 1 from row 2 resulting in a new row 2 of

$$\left(0, \binom{n+1}{m}\frac{1}{m+1}, \binom{n+1}{m-1}\frac{2}{m+1}, \dots, \binom{n+1}{m-k+2}\frac{k-1}{m+1}\right).$$

It follows that

$$\det A(n,m,k) = \binom{n}{m}(k-1)!\prod_{i=1}^{k-1}\frac{1}{m+i}\det D$$

where D is the $(k-1)\times(k-1)$ matrix given by

$$D = \begin{pmatrix} \binom{n+1}{m} & \binom{n+1}{m-1} & \cdots & \binom{n+1}{m-k+2} \\ \binom{n+1}{m+1} & \binom{n+1}{m} & \cdots & \binom{n+1}{m-k+3} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+1}{m+k-2} & \binom{n+1}{m+k-3} & \cdots & \binom{n+1}{m} \end{pmatrix},$$

that is, $D = A(n+1,m,k-1)$. Thus

$$\begin{aligned}
\det A(n,m,k) &= \binom{n}{m}(k-1)!\prod_{i=1}^{k-1}\frac{1}{m+i}\det A(n+1,m,k-1) \\
&= \binom{n}{m}(k-1)!\prod_{i=1}^{k-1}\frac{1}{m+i}\prod_{i=0}^{k-2}\binom{n+i+1}{m+i}\prod_{i=1}^{k-2}\frac{i!}{(n+1-m+i)^{k-i-1}} \\
&= \binom{n}{m}(k-1)!\prod_{i=1}^{k-1}\binom{n+i}{m+i}\frac{1}{(n-m+1)^{k-1}}\prod_{i=1}^{k-2}\frac{i!}{(n+1-m+i)^{k-i-1}} \\
&= \prod_{i=0}^{k-1}\binom{n+i}{m+i}\frac{(k-1)!}{(n-m+1)^{k-1}}\prod_{i=2}^{k-1}\frac{(i-1)!}{(n-m+i)^{k-i}} \\
&= \prod_{i=0}^{k-1}\binom{n+i}{m+i}\prod_{i=1}^{k-1}\frac{(i-1)!}{(n-m+i)^{k-i}}.
\end{aligned}$$

We used our inductive assumption in our expression of $\det A(n+1,m,k-1)$. The result follows. □

Proof of Corollary 1. Assume that $p \geq n+m-1$. Clearly A_k is invertible if and only if the $t \times t$ matrix $A_{t,t}$ is invertible for $t = 1, \ldots, k$. Now

$$A_{t,t} = \binom{n+m-2k}{n+m-2k} \begin{pmatrix} \binom{n+m-2k}{m-k} & \binom{n+m-2k}{m-k-1} & \cdots & \binom{n+m-2k}{m-k-t+1} \\ \binom{n+m-2k}{m-k+1} & \binom{n+m-2k}{m-k} & \cdots & \binom{n+m-2k}{m-k-t+2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+m-2k}{m-k+t-1} & \binom{n+m-2k}{m-k+i-2} & \cdots & \binom{n+m-2k}{m-k} \end{pmatrix}.$$

But by Lemma 2,

$$\det A_{t,t} = \prod_{j=0}^{t-1} \binom{n+m-2k+j}{m-k+j} \prod_{j=1}^{t-1} \frac{j!}{(n-k+j)^{t-j}}.$$

Thus the largest factor in the numerator is $n+m-2k+t-1$. Since

$$n+m-2k+t-1 \leq n+m-2k+k-1 = n+m-k-1 \leq n+m-2 < p,$$

$\det A_{t,t} \neq 0$ in K for $t = 1, \ldots, k$. We conclude A_k is invertible.

Since $\det A_k \neq 0$, and $(g-1)^{n+m-2k}(y_k) = (\det A_k)x_k$, the indecomposable KG-submodule KGy_k of $V_n \otimes V_m$ has dimension $n+m-2k+1$ and one-dimensional socle $KGx_k = Kx_k$. Since the socles are linearly independent over K, it follows that $KGy_1 + KGy_2 + \cdots + KGy_m$ is actually a direct sum. Because the dimension of $KGy_1 + KGy_2 + \cdots + KGy_m$ over K is $\sum_{k=1}^{m}(n+m-2k+1) = nm$,

$$V_n \otimes V_m = KGy_1 \oplus KGy_2 \oplus \cdots \oplus KGy_m.$$

□

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