A NEW ALGORITHM FOR DECOMPOSING MODULAR TENSOR PRODUCTS

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ABSTRACT. Let p be a prime and let J_r denote a full $r \times r$ Jordan block matrix with eigenvalue 1 over a field F of characteristic p. For positive integers r and s with $r \leq s$, the Jordan canonical form of the $rs \times rs$ matrix $J_r \otimes J_s$ has the form $J_{\lambda_1} \oplus J_{\lambda_2} \oplus \cdots \oplus J_{\lambda_r}$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. This decomposition determines a partition $\lambda(r,s,p) = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of rs but the values of the parts depend on r, s, and p. Shifting focus to the multiplicities of the parts of $\lambda(r,s,p)$, write

$$(\lambda_1, \lambda_2, \ldots, \lambda_r) = (\overbrace{\mu_1, \ldots, \mu_1}^{n_1}, \overbrace{\mu_2, \ldots, \mu_2}^{n_2}, \ldots, \overbrace{\mu_k, \ldots, \mu_k}^{n_k}) = (n_1 \cdot \mu_1, \ldots, n_k \cdot \mu_k),$$

where $\mu_1 > \mu_2 > \cdots > \mu_k > 0$. Then $c(r,s,p) = (n_1, \ldots, n_k)$ is a composition of r, from which $\lambda(r,s,p)$ can be computed easily. We present a new bottom-up algorithm for computing $c(r,s,p)$ directly from the base-p expansions for r and s.

1. Introduction

Determining the Jordan canonical form of the tensor product of Jordan blocks has many applications, for example, to the modular representations of finite cyclic p-groups, which we will explain shortly, as well as to algebraic groups [10, 11, 12], and to tilting modules.

As in the abstract, p is a prime number and r and s are positive integers with $r \leq s$. We now explain the connection of $\lambda(r,s,p)$ to the modular representation of a cyclic group G of order $q = p^\alpha$ where $r \leq s \leq q$. For a field F of characteristic p, it is well-known that there are exactly q isomorphism classes of indecomposable FG-modules and that such modules are cyclic and uniserial [1, p. 24–25]. Let $\{V_1, \ldots, V_q\}$ denote a complete set of representatives of these isomorphism classes with $\dim V_i = i$. Then if $r \leq s \leq q$, $V_r \otimes V_s \cong V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r}$ where $\lambda(r,s,p) = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. So decomposing $V_r \otimes V_s$ is equivalent to determining a particular partition $\lambda(r,s,p)$ of rs with r nonzero parts. Algorithms for computing $V_r \otimes V_s$, or equivalently $\lambda(r,s,p)$, start with Green [7] and include in order of appearance [18, 16, 17, 14, 15, 8, 9, 2]. As pointed out in [6, p. 216], these algorithms are either recursive top-down as in [7, 17, 14, 15, 8, 2], or related to binomial matrices as in [18, 16, 14, 9].

Since we will be dealing with $c(r,s,p)$ rather than $\lambda(r,s,p)$, we note that $\lambda(r,s,p)$ can be computed easily once $c(r,s,p)$ is known because

$$\mu_i = r + s - n_i - 2\sum_{j=1}^{i-1} n_j \tag{1}$$

by [6, Theorem 5].

Now we introduce our assumptions and notations.

The set-up. Write $r = \sum_{k=0}^{n} r_k p^k$ and $s = \sum_{k=0}^{n} s_k p^k$ where $0 \leq r_k, s_k < p$. For each integer i in the interval $[0,n]$, define $R_i = \sum_{k=0}^{i} r_k p^k$, $S_i = \sum_{k=0}^{i} s_k p^k$, $m_i = \min\{R_i, S_i\}$, and $M_i = \max\{R_i, S_i\}$. For i in the interval $[0,n]$, denote $c(m_i, M_i, p)$ by c_i, so c_i is a composition of m_i.

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Our algorithm starts by computing c_0, and then recursively computes c_i in terms of c_{i-1}, r_i, and s_i, arriving at $c_n = c(m_n, M_n, p) = c(R_n, S_n, p) = c(r, s, p)$. Thus the algorithm is bottom-up and makes essential use of the expressions of r and s in base p.

Why present another algorithm? Not for efficiency since the algorithm of Iima and Iwamatsu [9], which involves the computation of which of r determinants given by explicit formulas are divisible by p, meets that criterion well and is easy to program. A priori, one might expect algorithms for computing $\lambda(r, s, p)$ to be closely related to the base-p expansions of r and s, but this is not the case. For example, the popular algorithm of Renaud [17] essentially, in our language above, reduces the computation of $\lambda(R_n, S_n, p)$ to $\lambda(m_{n-1}, M_{n-1}, p)$, so that the link from $\lambda(R_n, S_n, p)$ to $\lambda(R_0, S_0, p)$ is implied but not specified explicitly. The algorithm presented here allows one for the first time to write down $c(r, s, p)$ by applying a series of decisions to the full base-p expansions of r and s.

To get a feel for our result we present a special case. Before doing so we explain certain notations that occur in it. If n is a nonnegative integer and $\alpha = (u_1, \dots, u_t)$ and $\beta = (v_1, \dots, v_\ell)$ are sequences of integers, then $rev(\alpha) = (u_t, \dots, u_1)$, $\alpha \oplus \beta = (u_1, \dots, u_t, v_1, \dots, v_\ell)$, $() \oplus \alpha = \alpha = \alpha \oplus ()$, and

$$n \cdot \alpha = \underbrace{\alpha \oplus \alpha \oplus \cdots \oplus \alpha}_{n}.$$

The special case has $R_i \le S_i$ and $R_i + S_i \le p^{i+1}$ for every i. Then $c_0 = r_0 \cdot \alpha_0 = r_0 \cdot (1)$. Assume $i > 0$ and let

$$\alpha_i = c_{i-1} \oplus (S_{i-1} - R_{i-1}) \oplus rev(c_{i-1}) \oplus (p^i - R_{i-1} - S_{i-1})$$

with the term $(S_{i-1} - R_{i-1})$ omitted if $S_{i-1} - R_{i-1} = 0$ and the term $(p^i - R_{i-1} - S_{i-1})$ omitted if $p^i - R_{i-1} - S_{i-1} = 0$. Then α_i is a composition of p^i and $c_i = r_i \cdot \alpha_i + c_{i-1}$. Hence

$$c(r, s, p) = c_n = r_n \cdot \alpha_n \oplus r_{n-1} \cdot \alpha_{n-1} \oplus \cdots \oplus r_0 \cdot \alpha_0$$

consists of r_n copies of a composition α_n of p^n, followed by r_{n-1} copies of a composition α_{n-1} of p^{n-1}, and so on.

The paper is organized as follows. After stating our result, Theorem 1, in Section 2, we review an earlier algorithm for computing $c(r, s, p)$ in Section 3, and then we use this in Section 4 to prove Theorem 1. Finally in Section 5, we apply our approach to give an alternate proof of a recent result of Glasby, Praeger, and Xia on the least part of $\lambda(r, r, p)$.

2. Results

Our set-up assumptions are in force.

Definition 1. For an integer i in $[0, n]$, we say there is an *overrun* at i if $R_i + S_i > p^{i+1}$.

Note that if $R_i \le S_i$ but $R_{i-1} > S_{i-1}$, then $R_i < S_i$ in fact. A similar observation applies to the situation $S_i \le R_i$ and $S_{i-1} > R_{i-1}$. This observation motivates our next definition.

Definition 2. We say there is a *switch* at an integer i in the interval $[1, n]$ if either $R_i < S_i$ and $R_{i-1} > S_{i-1}$ or $R_i > S_i$ and $R_{i-1} < S_{i-1}$, that is, if $(R_i - S_i)(R_{i-1} - R_{i-1}) < 0$.

The notations $rev(\alpha)$, $\alpha \oplus \beta$, and $n \cdot \alpha$, used in the next Theorem, were explained in Section 1.

Theorem 1. *Assume the set-up assumptions are in force. Then*

$$c_0 = \begin{cases} m_0 \cdot (1), & \text{if } r_0 + s_0 \le p, \\ (m_0 + M_0 - p) \oplus (p - M_0) \cdot (1), & \text{if } r_0 + s_0 > p. \end{cases}$$

Now assume that $i > 0$.

(1) *If there is no overrun at i, let*

$$\alpha_i = c_{i-1} \oplus (M_{i-1} - m_{i-1}) \oplus rev(c_{i-1}) \oplus (p^i - m_{i-1} - M_{i-1}).$$

Then

$$c_i = \begin{cases} \min\{r_i, s_i\} \cdot \alpha_i \oplus c_{i-1}, & \textit{if there is no switch at } i, \\ \min\{r_i, s_i\} \cdot \alpha_i \oplus c_{i-1} \oplus (M_{i-1} - m_{i-1}), & \textit{if there is a switch at } i. \end{cases}$$

(2) *If there is an overrun at i, let*

$$\alpha_i = (p^i - m_{i-1} - M_{i-1}) \oplus c_{i-1} \oplus (M_{i-1} - m_{i-1}) \oplus rev(c_{i-1})$$

and

$$\zeta_i = (m_i + M_i - p^{i+1}) \oplus (p - 1 - \max\{r_i, s_i\}) \cdot \alpha_i \oplus (p^i - m_{i-1} - M_{i-1}) \oplus c_{i-1}.$$

Then

$$c_i = \begin{cases} \zeta_i, & \textit{if there is no switch at } i, \\ \zeta_i \oplus (M_{i-1} - m_{i-1}), & \textit{if there is a switch at } i. \end{cases}$$

If $M_{i-1} - m_{i-1} = 0$ or $p^i - m_{i-1} - M_{i-1} = 0$, the corresponding term $(M_{i-1} - m_{i-1})$ or $(p^i - m_{i-1} - M_{i-1})$ is omitted.

Some comments

(1) Our expression for c_0 parallels the expression for $\lambda(r, s, p)$ when $1 \le r \le s \le p$ by Renaud in [17, Theorem 1].
(2) There are clear connections between the above theorem and Norman's [15, Theorem 4] even though Norman is concerned with a permutation $\pi(r, s)$ of $\{1, \ldots, r\}$ whereas we are concerned with a composition $c(r, s, p)$ of r.
(3) The α_i's are compositions of p^i because c_{i-1} is a composition of m_{i-1}.
(4) Note that when the term $(p^i - m_{i-1} - M_{i-1})$ occurs, either it is followed immediately by c_{i-1} or it immediately follows $rev(c_{i-1})$. Now $p^i - m_{i-1} - M_{i-1}$ will be negative if there is an overrun at $i - 1$. In this case to get just positive entries we can apply Lemma 1 to replace $(p^i - m_{i-1} - M_{i-1}) \oplus c(m_{i-1}, M_{i-1}, p)$ by $c(p^i - M_{i-1}, p^i - m_{i-1}, p)$ and $rev(c_{i-1}) \oplus (p^i - m_{i-1} - M_{i-1})$ by $rev(c(p^i - M_{i-1}, p^i - m_{i-1}, p))$ but then we lose the expression of c_i in terms of c_{i-1}.

Next we illustrate the theorem by computing $c(124, 282, 7) = c(2\times 49 + 3\times 7 + 5, 5\times 49 + 5\times 7 + 2, 7)$ which has overruns at 2 and 1 and a switch at 1. First $c_0 = c(2, 5, 7) = 2 \cdot (1) = (1, 1)$. Since there is an overrun at 1,

$$\alpha_1 = (7 - 2 - 5) \oplus c_0 \oplus (5 - 2) \oplus rev(c_0) = () \oplus (1, 1) \oplus (3) \oplus (1, 1) = (1, 1, 3, 1, 1).$$

Since there is an overrun and at switch at 1,

$$\begin{aligned} c_1 &= c(3\times 7 + 5, 5\times 7 + 2, 7) \\ &= (26 + 37 - 49) \oplus 1 \cdot \alpha_1 \oplus (7 - 2 - 5) \oplus c_0 \oplus (5 - 2) \\ &= (14) \oplus (1, 1, 3, 1, 1) \oplus () \oplus (1, 1) \oplus (3) \\ &= (14, 1, 1, 3, 1, 1, 1, 1, 3). \end{aligned}$$

Since there is an overrun at 2,

$$\begin{aligned} \alpha_2 &= (49 - 26 - 37) \oplus c_1 \oplus (37 - 26) \oplus rev(c_1) \\ &= (-14) \oplus (14, 1, 1, 3, 1, 1, 1, 1, 3) \oplus (11) \oplus (3, 1, 1, 1, 1, 3, 1, 1, 14) \\ &= (1, 1, 3, 1, 1, 1, 1, 3, 11, 3, 1, 1, 1, 1, 3, 1, 1, 14) \end{aligned}$$

and

$$\begin{aligned}
c_2 &= c(124, 282, 7)\\
&= (124 + 282 - 343) \oplus 1 \cdot \alpha_2 \oplus (49 - 26 - 37) \oplus c_1\\
&= (63) \oplus (1, 1, 3, 1, 1, 1, 1, 3, 11, 3, 1, 1, 1, 1, 3, 1, 1, 14) \oplus (-14) \oplus (14, 1, 1, 3, 1, 1, 1, 1, 3)\\
&= (63, 1, 1, 3, 1, 1, 1, 1, 3, 11, 3, 1, 1, 1, 1, 3, 1, 1, 14, 1, 1, 3, 1, 1, 1, 1, 3).
\end{aligned}$$

3. An Algorithm for Computing $c(r, s, p)$

To prove our results we need to use an existing algorithm for computing $c(r, s, p)$ or equivalently $\lambda(r, s, p)$. The present author gave a recursive algorithm in [2] for computing $V_r \otimes V_s$ as a sum of indecomposables and rephrased it as an algorithm for computing $c(r, s, p)$ in [3] which we now describe.

For any nonnegative integer s, define $c(0, s, p) = ()$. Now assume $0 < r \leq s$. Let k be the unique nonnegative integer k such that $p^k \leq s < p^{k+1}$, so $k = \lfloor \log_p(s) \rfloor$. Write $s = s_k p^k + S_{k-1}$ with $1 \leq s_k < p$ and $0 \leq S_{k-1} < p^k$ and $r = r_k p^k + R_{k-1}$ where $0 \leq r_k < p$ and $0 \leq R_{k-1} < p^k$. The following six cases are exhaustive and mutually exclusive.

Case 1 If $(r + s > p^{k+1})$:
Then $c(r, s, p) = (r + s - p^{k+1}) \oplus c(p^{k+1} - s, p^{k+1} - r, p)$.
Case 2 else if $(r + s \leq p^{k+1}) \wedge (R_{k-1} + S_{k-1} > p^k)$:
Here $r_k + s_k \leq p-2$. Then $c(r, s, p) = (R_{k-1} + S_{k-1} - p^k) \oplus c((r_k + s_k + 1)p^k - s, (r_k + s_k + 1)p^k - r, p)$.
Case 3 else if $(r + s \leq p^{k+1}) \wedge (1 \leq R_{k-1} + S_{k-1} \leq p^k) \wedge (r_k > 0)$:
Let $m_{k-1} = \min\{R_{k-1}, S_{k-1}\}$, $M_{k-1} = \max\{R_{k-1}, S_{k-1}\}$, and

$$u = c(m_{k-1}, M_{k-1}, p) \oplus (M_{k-1} - m_{k-1}) \oplus rev(c(m_{k-1}, M_{k-1}, p))$$

with the term $(M_{k-1} - m_{k-1})$ omitted if $m_{k-1} = M_{k-1}$.
Then $c(r, s, p) = u \oplus c((r_k + s_k)p^k - s, (r_k + s_k)p^k - r, p)$.
Case 4 else if $(r + s \leq p^{k+1}) \wedge (1 \leq R_{k-1} + S_{k-1} \leq p^k) \wedge (r_k = 0) \wedge (S_{k-1} = 0)$:
In this case $r = R_{k-1}$ and $s = s_k p^k$. Then $c(r, s, p) = (R_{k-1})$.
Case 5 else if $(r + s \leq p^{k+1}) \wedge (1 \leq R_{k-1} + S_{k-1} \leq p^k) \wedge (r_k = 0) \wedge (S_{k-1} > 0)$:
In this case $r = R_{k-1}$. Then $c(r, s, p) = c(R_{k-1}, s_k p^k + S_{k-1}, p) = rev(c(R_{k-1}, s_k p^k - S_{k-1}, p))$.
Case 6 else :
Here $(r + s \leq p^{k+1}) \wedge (R_{k-1} = 0) \wedge (S_{k-1} = 0)$. So $r = r_k p^k$ with $r_k > 0$ and $s = s_k p^k$. Then $c(r, s, p) = c(r_k p^k, s_k p^k, p) = (p^k) \oplus c((r_k - 1)p^k, (s_k - 1)p^k, p)$.

Note that Case 6 can be rewritten non-recursively as $c(r, s, p) = c(r_k p^k, s_k p^k, p) = r_k \cdot (p^k)$.

We end this section with an easy consequence of Case 1 of the algorithm.

Lemma 1. *Suppose that r and s are integers with $1 \leq r \leq s < p^k$. If $r + s > p^k$, $c(r, s, p) = (r + s - p^k) \oplus c(p^k - s, p^k - r, p)$, whereas if $r + s \leq p^k$, $c(p^k - s, p^k - r, p) = (p^k - r - s) \oplus c(r, s, p)$. We can combine the two cases as*

$$c(p^k - s, p^k - r, p) = (p^k - r - s) \oplus c(r, s, p)$$

where $p^k - r - s < 0$ in the first assertion and nonnegative in the second.

Proof. The first assertion follows directly from Case 1. If $r + s = p^k$, then the second assertion states that $c(r, s, p) = c(r, s, p)$. Assume that $r + s < p^k$. Then $(p^k - s) + (p^k - r) > p^k$, and $c(p^k - s, p^k - r, p) = (p^k - s + p^k - r - p^k) \oplus c(r, s, p)$ by Case 1. □

Note that a version of this result for $\lambda(r, s, p)$ occurs as Proposition 14 in [6].

4. Proof

We record a property of $c(r,s,p)$ that follows from a recent important result of Glasby, Praeger, and Xia [5, Theorem 4].

Theorem 2 (GPX). *Suppose that $1 \le r \le p^n$. Then $c(r,s,p) = c(r, s+p^n, p)$ for every integer $s \ge r$.*

Crucial to this paper is the partial subperiodic behavior recorded in the next lemma. We proved a slightly more general form of this in [4, Theorem 3].

Lemma 2. *Let r be a positive integer with $r \le \lfloor p^{k+1}/2 \rfloor$. Let s and s' be any two integers in the interval $[r, p^{k+1}-r]$ such that $s \equiv s' \mod p^k$. Then $c(r,s,p) = c(r,s',p)$.*

Proof. Let r be the least integer for which this result is false and let s and s' be integers in the interval $[r, p^{k+1}-r]$ such that $c(r,s,p) \ne c(r,s',p)$. When $r \le p^k$, which is the only possibility when $p=2$, the result follows from Theorem 2. Hence $r > p^k$. Assuming that $s' \le s$, write $r = r_k p^k + R_{k-1}$, $s' = s'_k p^k + S_{k-1}$, and $s = s_k p^k + S_{k-1}$ where $1 \le r_k \le s'_k \le s_k \le p-1$ and $0 \le R_{k-1}, S_{k-1} < p^k$. Exactly one of Cases 2, 3 and 6 applies to the computation of both $c(r,s',p)$ and $c(r,s,p)$.

In Case 2, since $R_{k-1} + S_{k-1} > p^k$, $R_{k-1} > 0$ and $S_{k-1} > 0$, and

$$\begin{aligned} c(r,s,p) &= (R_{k-1}+S_{k-1}-p^k) \oplus c((r_k+s_k+1)p^k - s, (r_k+s_k+1)p^k - r, p) \\ &= (R_{k-1}+S_{k-1}-p^k) \oplus c((r_k+s_k+1)p^k - (s_k p^k + S_{k-1}), (r_k+s_k+1)p^k - (r_k p^k + R_{k-1}), p) \\ &= (R_{k-1}+S_{k-1}-p^k) \oplus c(r_k p^k + p^k - S_{k-1}, s_k p^k + p^k - R_{k-1}, p) \end{aligned}$$

and

$$\begin{aligned} c(r,s',p) &= (R_{k-1}+S_{k-1}-p^k) \oplus c((r_k+s'_k+1)p^k - s', (r_k+s'_k+1)p^k - r, p) \\ &= (R_{k-1}+S_{k-1}-p^k) \oplus c(r_k p^k + p^k - S_{k-1}, s'_k p^k + p^k - R_{k-1}, p). \end{aligned}$$

Since $(r_k p^k + p^k - S_{k-1}) + (s_k p^k + p^k - R_{k-1}) < r + s \le p^{k+1}$, the minimality of r implies that

$$c(r_k p^k + p^k - S_{k-1}, s_k p^k + p^k - R_{k-1}, p) = c(r_k p^k + p^k - S_{k-1}, s'_k p^k + p^k - R_{k-1}, p)$$

which gives $c(r,s,p) = c(r,s',p)$ — a contradiction!

In Case 3, where $1 \le R_{k-1} + S_{k-1} \le p^k$ and $r_k > 0$,

$$c(r,s,p) = u \oplus c((r_k+s_k)p^k - s, (r_k+s_k)p^k - r, p) = u \oplus c((r_k-1)p^k + p^k - S_{k-1}, (s_k-1)p^k + p^k - R_{k-1}, p)$$

and

$$c(r,s',p) = u \oplus c((r_k+s'_k)p^k - s', (r_k+s'_k)p^k - r, p) = u \oplus c((r_k-1)p^k + p^k - S_{k-1}, (s'_k-1)p^k + p^k - R_{k-1}, p)$$

where u is defined as in Case 3. Since

$$((r_k-1)p^k + p^k - S_{k-1}) + ((s_k-1)p^k + p^k - R_{k-1}) < (r_k+s_k-2)p^k + 2p^k = (r_k+b_k)p^k < p^{k+1},$$

the minimality of r implies $c((r_k-1)p^k + p^k - S_{k-1}, (s_k-1)p^k + p^k - R_{k-1}, p) = c((r_k-1)p^k + p^k - S_{k-1}, (s_k,-1)p^k + p^k - R_{k-1}, p)$ which gives $c(r,s,p) = c(r,s',p)$ — a contradiction!

In Case 6, $R_{k-1} = S_{k-1} = 0$, and

$$c(r,s,p) = c(r_k p^k, s_k p^k, p) = r_k \cdot (p^k) = c(r_k p^k, s'_k p^k, p) = c(r,s',p) \text{ — a contradiction!}$$

□

Corollary 1. *Suppose that $r = r_k p^k + R_{k-1}$ and $s = s_k p^k + S_{k-1}$ where $1 \le r \le s < r+s \le p^{k+1}$, $0 \le r_k \le s_k < p$ and $0 \le R_{k-1}, S_{k-1} < p^k$. Then*

$$c(r,s,p) = \begin{cases} c(r, r_k p^k + S_{k-1}, p), & \text{if } R_{k-1} \le S_{k-1}, \\ c(r, (r_k+1)p^k + S_{k-1}, p), & \text{if } R_{k-1} > S_{k-1}. \end{cases}$$

Proof. Let s' be the least integer such that $s' \geq r$ and $s' \equiv s \mod p^k$. Then $s' = r_k p^k + S_{k-1}$ if $R_{k-1} \leq S_{k-1}$ and $s' = (r_k+1)p^k + S_{k-1}$ if $R_{k-1} > S_{k-1}$. By Lemma 2, $c(r,s,p) = c(r,s',p)$. □

Lemma 3. *If $i > 0$ and there is neither an overrun nor a switch at i, then*

$$c_i = \min\{r_i, s_i\} \cdot \alpha_i \oplus c_{i-1}$$

where

$$\alpha_i = c_{i-1} \oplus (M_{i-1} - m_{i-1}) \oplus rev(c_{i-1}) \oplus (p^i - m_{i-1} - M_{i-1})$$

with the term $(M_{i-1} - m_{i-1})$ omitted if $m_{i-1} = M_{i-1}$ and the term $(p^i - m_{i-1} - M_{i-1})$ omitted if $m_{i-1} + M_{i-1} = p^i$.

Proof. Without loss of generality we can assume that $m_i = R_i$, so $M_i = S_i$. Since there is no overrun at i, $R_i + S_i = m_i + M_i \leq p^{i+1}$, and since there is no switch at i, $R_{i-1} \leq S_{i-1}$. Hence we must prove that $c(R_i, S_i, p) = r_i \cdot \alpha_i \oplus c(R_{i-1}, S_{i-1}, p)$.

Assume that $S_{i-1} = 0$. Then $R_{i-1} = 0$ and $\alpha_i = () \oplus () \oplus () \oplus (p^i) = (p^i)$. By Case 6, $c(r,s,p) = c(r_i p^i, s_i p^i, p) = r_i \cdot (p^i)$. But $r_i \cdot (p^i) = r_i \cdot \alpha_i \oplus () = r_i \cdot \alpha_i \oplus c(R_{i-1}, S_{i-1}, p)$. Thus the result holds in this case.

Now assume that $S_{i-1} > 0$. Denote $c(R_{i-1}, S_{i-1}, p) \oplus (S_{i-1} - R_{i-1}) \oplus rev(c(R_{i-1}, S_{i-1}, p))$ by ν where the term $(S_{i-1} - R_{i-1})$ is omitted if $R_{i-1} = S_{i-1}$. Then

$$\begin{aligned} c(R_i, S_i, p) &= c(r_i p^i + R_{i-1}, s_i p^i + S_{i-1}, p) \\ &= c(r_i p^i + R_{i-1}, r_i p^i + S_{i-1}, p), && \text{by Corollary 1,} \\ &= \nu \oplus c(2r_i p^i - (r_i p^i + S_{i-1}), 2r_i p^i - (r_i p^i + R_{i-1}), p), && \text{by Case 3,} \\ &= \nu \oplus c((r_i - 1)p^i + p^i - S_{i-1}, (r_i - 1)p^i + p^i - R_{i-1}, p). \end{aligned}$$

Now if $(p^i - S_{i-1}) + (p^i - R_{i-1}) > p^i$, which occurs precisely when $R_{i-1} + S_{i-1} < p^i$, then by Case 2

$$c((r_i-1)p^i + p^i - S_{i-1}, (r_i-1)p^i + p^i - R_{i-1}, p) = (p^i - R_{i-1} - S_{i-1}) \oplus c((r_i-1)p^i + R_{i-1}, (r_i-1)p^i + S_{i-1}, p).$$

On the other hand if $R_{i-1} + S_{i-1} = p^i$,

$$c((r_i - 1)p^i + p^i - S_{i-1}, (r_i - 1)p^i + p^i - R_{i-1}, p) = c((r_i - 1)p^i + R_{i-1}, (r_i - 1)p^i + S_{i-1}, p).$$

Thus $c(r_i p^i + R_{i-1}, r_i p^i + S_{i-1}, p) = \alpha_i \oplus c((r_i - 1)p^i + R_{i-1}, (r_i - 1)p^i + S_{i-1}, p)$, and the result follows easily by induction.

□

For our next proof, we need the following result which is a corollary of a result of J. A. Green.

Lemma 4. *If $1 \leq r \leq s < r + s \leq p^k$, then $c(r, p^k - s, p) = rev(c(r,s,p))$.*

Proof. By [7, (2.5a)], if $\lambda(r,s,p) = (\lambda_1, \ldots, \lambda_r)$, then $\lambda(r, p^k - s, p) = (p^k - \lambda_r, \ldots, p^k - \lambda_1)$. Our result is an immediate consequence of this. □

Lemma 5. *If $i > 0$ and there is a switch but no overrun at i, then*

$$c_i = \min\{r_i, s_i\} \cdot \alpha_i \oplus c_{i-1} \oplus (M_{i-1} - m_{i-1})$$

where

$$\alpha_i = c_{i-1} \oplus (M_{i-1} - m_{i-1}) \oplus rev(c_{i-1}) \oplus (p^i - m_{i-1} - M_{i-1})$$

with the term $(p^i - m_{i-1} - M_{i-1})$ omitted if $m_{i-1} + M_{i-1} = p^i$.

Proof. Again without loss of generality we can assume that $m_i = R_i$, so $M_i = S_i$. Since there is no overrun at i, $R_i + S_i = m_i + M_i \le p^{i+1}$, and since there is a switch at i, $R_{i-1} > S_{i-1}$ (and $R_i < S_i$), so $m_{i-1} = S_{i-1}$ and $M_{i-1} = R_{i-1}$. Hence we must prove that $c(R_i, S_i, p) = r_i \cdot \alpha_i \oplus c(S_{i-1}, R_{i-1}, p) \oplus (R_{i-1} - S_{i-1})$. Then by Corollary 1,

$$c(R_i, S_i, p) = c(r_i p^i + R_{i-1}, s_i p^i + S_{i-1}, p) = c(r_i p^i + R_{i-1}, (r_i + 1)p^i + S_{i-1}, p).$$

Since the proof that $c(r_i p^i + R_{i-1}, (r_i+1)p^i + S_{i-1}, p) = r_i \cdot \alpha_i \oplus c(R_{i-1}, p^i + S_{i-1}, p)$ is similar to the proof of Lemma 3, we omit it. It remains to show that $c(R_{i-1}, p^i + S_{i-1}, p) = c(S_{i-1}, R_{i-1}, p) \oplus (R_{i-1} - S_{i-1})$. Assume for now there is no overrun at $i-1$, so $R_{i-1} + S_{i-1} \le p^i$. Then

$$\begin{aligned}
c(R_{i-1}, p^i + S_{i-1}, p) &= rev(c(R_{i-1}, p^i - S_{i-1}, p)), && \text{by Case 5 since } R_{i-1} + S_{i-1} \le p^i,\\
&= rev((R_{i-1} - S_{i-1}) \oplus c(p^i - (p^i - S_{i-1}), p^i - R_{i-1}, p)), && \text{by Case 1}\\
&= rev((R_{i-1} - S_{i-1}) \oplus c(S_{i-1}, p^i - R_{i-1}, p))\\
&= rev((R_{i-1} - S_{i-1}) \oplus rev(c(S_{i-1}, R_{i-1}, p))), && \text{by Lemma 4,}\\
&= c(S_{i-1}, R_{i-1}, p) \oplus (R_{i-1} - S_{i-1}).
\end{aligned}$$

Now consider the case of an overrun at $i-1$. Then

$$\begin{aligned}
c(R_{i-1}, p^i + S_{i-1}, p) &= (R_{i-1} + S_{i-1} - p^i) \oplus c(2p^i - (p^i + S_{i-1}), 2p^i - R_{i-1}, p), && \text{by Case 2,}\\
&= (R_{i-1} + S_{i-1} - p^i) \oplus c(p^i - S_{i-1}, p^i + p^i - R_{i-1}, p).
\end{aligned}$$

By the calculation of the previous paragraph since $(p^i - S_{i-1}) + (p^i - R_{i-1}) < p^i$ and $(p^i - R_{i-1}) < (p^i - S_{i-1})$,

$$\begin{aligned}
c(p^i - S_{i-1}, p^i + p^i - R_{i-1}, p) &= c(p^i - R_{i-1}, p^i - S_{i-1}, p) \oplus ((p^i - S_{i-1}) - (p^i - R_{i-1}))\\
&= c(p^i - R_{i-1}, p^i - S_{i-1}, p) \oplus (R_{i-1} - S_{i-1}).
\end{aligned}$$

Thus

$$\begin{aligned}
c(R_{i-1}, p^i + S_{i-1}, p) &= (R_{i-1} + S_{i-1} - p^i) \oplus c(p^i - R_{i-1}, p^i - S_{i-1}, p) \oplus (R_{i-1} - S_{i-1})\\
&= c(S_{i-1}, R_{i-1}, p) \oplus (R_{i-1} - S_{i-1})
\end{aligned}$$

by Case 1. □

Lemma 6. *If $i > 0$ and there is an overrun but no switch at i, then*

$$c_i = (m_i + M_i - p^{i+1}) \oplus (p - 1 - \max\{r_i, s_i\}) \cdot \alpha_i \oplus (p^i - m_{i-1} - M_{i-1}) \oplus c_{i-1}$$

where

$$\alpha_i = (p^i - m_{i-1} - M_{i-1}) \oplus c_{i-1} \oplus (M_{i-1} - m_{i-1}) \oplus rev(c_{i-1})$$

where the term $(p^i - m_{i-1} - M_{i-1})$ is omitted if $m_{i-1} + M_{i-1} = p^i$ and the term $(M_{i-1} - m_{i-1})$ is omitted if $m_{i-1} = M_{i-1}$.

Proof. Without loss of generality we can assume that $R_i \le S_i$. Since there is an overrun but no switch at i, $R_i + S_i > p^{i+1}$ and $R_{i-1} \le S_{i-1}$. We must show that

$$c(R_i, S_i, p) = (R_i + S_i - p^{i+1}) \oplus (p - 1 - s_i) \cdot \alpha_i \oplus (p^i - R_{i-1} - S_{i-1}) \oplus c(R_{i-1}, S_{i-1}, p).$$

Now

$$\begin{aligned}
c(R_i, S_i, p) &= (R_i + S_i - p^{i+1}) \oplus c(p^{i+1} - S_i, p^{i+1} - R_i, p), && \text{by Case 1}\\
&= (R_i + S_i - p^{i+1}) \oplus c(p^{i+1} - (s_i p^i + S_{i-1}), p^{i+1} - (r_i p^i + R_{i-1}), p).
\end{aligned}$$

Assume for now that $S_{i-1} > 0$. Then

$$\begin{aligned}
c(R_i, S_i, p) &= (R_i + S_i - p^{i+1}) \oplus c((p - s_i - 1)p^i + p^i - S_{i-1}, p^{i+1} - (r_i p^i + R_{i-1}), p)\\
&= (R_i + S_i - p^{i+1}) \oplus c((p - s_i - 1)p^i + p^i - S_{i-1}, (p - s_i - 1)p^i + p^i - R_{i-1}, p)
\end{aligned}$$

by Corollary 1. Hence we must prove

$$c((p-s_i-1)p^i+p^i-S_{i-1},(p-s_i-1)p^i+p^i-R_{i-1},p)=(p-s_i-1)\cdot\alpha_i\oplus(p^i-R_{i-1}-S_{i-1})\oplus c_{i-1}.$$

Now Lemma 3 with R_{i-1} replaced by p^i-S_{i-1} and S_{i-1} replaced by p^i-R_{i-1} implies that

$$c((p-s_i-1)p^i+p^i-S_{i-1},(p-s_i-1)p^i+p^i-R_{i-1},p)=(p-s_i-1)\cdot\nu_i\oplus c(p^i-S_{i-1},p^i-R_{i-1},p)$$

where

$$\begin{aligned}\nu_i&=c(p^i-S_{i-1},p^i-R_{i-1},p)\oplus(p^i-R_{i-1}-(p^i-S_{i-1}))\\&\quad\oplus rev(c(p^i-S_{i-1},p^i-R_{i-1},p))\oplus(p^i-(p^i-S_{i-1})-(p^i-R_{i-1}))\\&=c(p^i-S_{i-1},p^i-R_{i-1},p)\oplus(S_{i-1}-R_{i-1})\oplus rev(c(p^i-S_{i-1},p^i-R_{i-1},p))\oplus(R_{i-1}+S_{i-1}-p^i)\end{aligned}$$

But $c(p^i-S_{i-1},p^i-R_{i-1},p)=(p^i-R_{i-1}-S_{i-1})\oplus c(R_{i-1},S_{i-1},p)=(p^i-R_{i-1}-S_{i-1})\oplus c_{i-1}$ by Lemma 1. Thus

$$\begin{aligned}\nu_i&=(p^i-R_{i-1}-S_{i-1})\oplus c_{i-1}\oplus(S_{i-1}-R_{i-1})\oplus rev(c_{i-1})\oplus(p^i-R_{i-1}-S_{i-1})\oplus(R_{i-1}+S_{i-1}-p^i)\\&=(p^i-R_{i-1}-S_{i-1})\oplus c_{i-1}\oplus(S_{i-1}-R_{i-1})\oplus rev(c_{i-1})\\&=\alpha_i.\end{aligned}$$

Hence

$$\begin{aligned}c(R_i,S_i,p)&=(R_i+S_i-p^{i+1})\oplus(p-s_i-1)\cdot\alpha_i\oplus c(p^i-S_{i-1},p^i-R_{i-1},p)\\&=(R_i+S_i-p^{i+1})\oplus(p-s_i-1)\cdot\alpha_i\oplus(p^i-R_{i-1}-S_{i-1})\oplus c(R_{i-1},S_{i-1},p)\\&=(R_i+S_i-p^{i+1})\oplus(p-s_i-1)\cdot\alpha_i\oplus(p^i-R_{i-1}-S_{i-1})\oplus c_{i-1}.\end{aligned}$$

Now assume that $S_{i-1}=0$. Hence $R_{i-1}=0$. Then

$$\begin{aligned}c(R_i,S_i,p)&=(R_i+S_i-p^{i+1})\oplus c(p^{i+1}-S_i,p^{i+1}-R_i,p), &&\text{by Case 1,}\\&=(R_i+S_i-p^{i+1})\oplus c(p^{i+1}-(s_ip^i+S_{i-1}),p^{i+1}-(r_ip^i+R_{i-1}),p)\\&=(R_i+S_i-p^{i+1})\oplus c((p-s_i)p^i,(p-r_i)p^i,p)\\&=(R_i+S_i-p^{i+1})\oplus(p-s_i)\cdot(p^i), &&\text{by Case 6.}\end{aligned}$$

But in this case

$$\alpha_i=(p^i-R_{i-1}-S_{i-1})\oplus c_{i-1}\oplus(M_{i-1}-m_{i-1})\oplus rev(c_{i-1})=(p^i)\oplus()\oplus()\oplus()=(p^i).$$

Thus

$$\begin{aligned}c(R_i,S_i,p)&=(R_i+S_i-p^{i+1})\oplus(p-1-s_i)\cdot(p^i)\oplus(p^i)\\&=(R_i+S_i-p^{i+1})\oplus(p-1-s_i)\cdot\alpha_i\oplus(p^i)\oplus()\\&=(R_i+S_i-p^{i+1})\oplus(p-1-s_i)\cdot\alpha_i\oplus(p^i-R_{i-1}-S_{i-1})\oplus c_{i-1},\end{aligned}$$

which is what we need to show. □

Lemma 7. *If $i>0$ and there is an overrun and a switch at i, then*

$$c_i=(m_i+M_i-p^{i+1})\oplus(p-1-\max\{r_i,s_i\})\cdot\alpha_i\oplus(p^i-m_{i-1}-M_{i-1})\oplus c_{i-1}\oplus(M_{i-1}-m_{i-1})$$

where

$$\alpha_i=(p^i-m_{i-1}-M_{i-1})\oplus c_{i-1}\oplus(M_{i-1}-m_{i-1})\oplus rev(c_{i-1})$$

where the term $(p^i-m_{i-1}-M_{i-1})$ is omitted if $m_{i-1}+M_{i-1}=p^i$.

Proof. The proof is really a combination of the proofs of Lemmas 6 and 5 but some details are trickier. Without loss of generality we can assume that $R_i\le S_i$. Since there is an overrun and switch at i, $R_i+S_i>p^{i+1}$, and $R_{i-1}>S_{i-1}$(and $R_i<S_i$). We must show that

$$c(R_i,S_i,p)=(R_i+S_i-p^{i+1})\oplus(p-1-s_i)\cdot\alpha_i\oplus(p^i-R_{i-1}-S_{i-1})\oplus c(S_{i-1},R_{i-1},p)\oplus(R_{i-1}-S_{i-1}).$$

Now

$$\begin{aligned} c(R_i, S_i, p) &= (R_i + S_i - p^{i+1}) \oplus c(p^{i+1} - S_i, p^{i+1} - R_i, p), && \text{by Case 1} \\ &= (R_i + S_i - p^{i+1}) \oplus c(p^{i+1} - (s_i p^i + S_{i-1}), p^{i+1} - (r_i p^i + R_{i-1}), p). \end{aligned}$$

Assume for now that $S_{i-1} > 0$. Then

$$\begin{aligned} c(R_i, S_i, p) &= (R_i + S_i - p^{i+1}) \oplus c((p - b_i - 1)p^i + p^i - S_{i-1}, p^{i+1} - (r_i p^i + R_{i-1}), p) \\ &= (R_i + S_i - p^{i+1}) \oplus c((p - s_i - 1)p^i + p^i - S_{i-1}, (p - s_i)p^i + p^i - R_{i-1}, p) \end{aligned}$$

by Corollary 1. Hence we must prove

$$\begin{aligned} &c((p - b_i - 1)p^i + p^i - S_{i-1}, (p - b_i)p^i + p^i - R_{i-1}, p) \\ &\qquad = (p - s_i - 1) \cdot \alpha_i \oplus (p^i - R_{i-1} - S_{i-1}) \oplus c(S_{i-1}, R_{i-1}, p) \oplus (R_{i-1} - S_{i-1}). \end{aligned}$$

By Lemma 5, with R_{i-1} replaced by $p^i - S_{i-1}$ and S_{i-1} replaced by $p^i - R_{i-1}$, we get

$$\begin{aligned} &c((p - s_i - 1)p^i + p^i - S_{i-1}, (p - s_i)p^i + p^i - R_{i-1}, p) \\ &\quad = (p - s_i - 1) \cdot \nu_i \oplus c(p^i - R_{i-1}, p^i - S_{i-1}, p) \oplus (p^i - S_{i-1} - (p^i - R_{i-1})) \\ &\quad = (p - s_i - 1) \cdot \nu_i \oplus c(p^i - R_{i-1}, p^i - S_{i-1}, p) \oplus (R_{i-1} - S_{i-1}) \end{aligned}$$

where

$$\begin{aligned} \nu_i &= c(p^i - R_{i-1}, p^i - S_{i-1}, p) \oplus (p^i - S_{i-1} - (p^i - R_{i-1})) \\ &\quad \oplus rev(c(p^i - R_{i-1}, p^i - S_{i-1}, p)) \oplus (p^i - (p^i - R_{i-1}) - (p^i - S_{i-1})) \\ &= c(p^i - R_{i-1}, p^i - S_{i-1}, p) \oplus (R_{i-1} - S_{i-1}) \oplus rev(c(p^i - R_{i-1}, p^i - S_{i-1}, p)) \oplus (R_{i-1} + S_{i-1} - p^i). \end{aligned}$$

As in the proof of Lemma 6, we have $\nu_i = \alpha_i$. Hence

$$\begin{aligned} c(R_i, S_i, p) &= (R_i + S_i - p^{i+1}) \oplus (p - s_i - 1) \cdot \alpha_i \oplus c(p^i - R_{i-1}, p^i - S_{i-1}, p) \oplus (R_{i-1} - S_{i-1}) \\ &= (R_i + S_i - p^{i+1}) \oplus (p - s_i - 1) \cdot \alpha_i \oplus (p^i - R_{i-1} - S_{i-1}) \oplus c(S_{i-1}, R_{i-1}, p) \oplus (R_{i-1} - S_{i-1}) \\ &= (R_i + S_i - p^{i+1}) \oplus (p - s_i - 1) \cdot \alpha_i \oplus (p^i - R_{i-1} - S_{i-1}) \oplus c_{i-1} \oplus (R_{i-1} - S_{i-1}). \end{aligned}$$

Now assume that $S_{i-1} = 0$. Then $R_{i-1} > 0$ and

$$\begin{aligned} c(R_i, S_i, p) &= (R_i + S_i - p^{i+1}) \oplus c((p - s_i - 1)p^i + p^i - S_{i-1}, p^{i+1} - (r_i p^i + R_{i-1}), p) \\ &= (R_i + S_i - p^{i+1}) \oplus c((p - s_i)p^i, (p - s_i + 1)p^i + p^i - R_{i-1}, p) \\ &= (R_i + S_i - p^{i+1}) \oplus (p - s_i) \cdot \alpha_i \oplus c(0, p^i - R_{i-1}, p) \end{aligned}$$

by Corollary 1. But in this case

$\alpha_i = (p^i - R_{i-1} - 0) \oplus c_{i-1} \oplus (R_{i-1} - 0) \oplus rev(c_{i-1}) = (p^i - R_{i-1}) \oplus () \oplus (R_{i-1}) \oplus () = (p^i - R_{i-1}) \oplus (R_{i-1})$.

Thus

$$\begin{aligned} c(R_i, S_i, p) &= (R_i + S_i - p^{i+1}) \oplus (p - s_i) \cdot (p^i - R_{i-1}, R_{i-1}) \\ &= (R_i + S_i - p^{i+1}) \oplus (p - 1 - s_i) \cdot \alpha_i \oplus (p^i - R_{i-1}, R_{i-1}) \\ &= (R_i + S_i - p^{i+1}) \oplus (p - 1 - s_i) \cdot \alpha_i \oplus (p^i - R_{i-1} - S_{i-1}) \oplus () \oplus (R_{i-1} - S_{i-1}) \\ &= (R_i + S_i - p^{i+1}) \oplus (p - 1 - s_i) \cdot \alpha_i \oplus (p^i - R_{i-1} - S_{i-1}) \oplus c_{i-1} \oplus (R_{i-1} - S_{i-1}) \end{aligned}$$

which is what we need to show. □

Proof of Theorem 1. The result for c_0 follows from Case 6 with $k = 0$ when $r_0 + s_0 \le p$. When $r_0 + s_0 > p$,

$$c_0 = c(m_0, M_0, p) = (m_0 + M_0 - p) \oplus c(p - M_0, p - m_0, p) = (m_0 + M_0 - p) \oplus (p - M_0) \cdot (1)$$

by Case 1 followed by Case 6. The result for $i > 0$ is proved by Lemmas 3, 5, 6, and 7. □

5. Application

We use Theorem 1 to give an alternate proof of the following result which was proved in [6, Theorem 13] using results of [9] and divisibility properties of binomial coefficients.

Theorem 3. *Let* $r = \sum_{i=0}^{n} r_i p^i$ *be the expansion of a positive integer* r *in base* p *and* $t = \min\{i \mid r_i > 0\}$*. Then the least part of* $\lambda(r, r, p) = (n_1 \cdot \mu_1, \ldots, n_k \cdot \mu_k)$ *is* p^t *and it occurs with multiplicity* p^t*, that is,* $\mu_k = p^t = n_k$*.*

Proof. Note that there are no switches in the computation of $c(r, r, p) = (n_1, \ldots, n_k)$. Therefore $c(r, r, p) = \psi_i \oplus c_i$ for every i in the interval $[0, n]$ by Theorem 1. In particular $c(r, r, p) = \psi_t \oplus c_t$ where $c_t = c(r_t p^t, r_t p^t, p)$. First we show that n_k, the multiplicity of the least part μ_k of $\lambda(r, r, p)$, is p^t. When $t = 0$, $n_k = 1 = p^t$ by the base case of Theorem 1. If $t > 0$ and there is no overrun at t, then by Theorem 1

$$\alpha_t = c_{t-1} \oplus (M_{t-1} - m_{t-1}) \oplus rev(c_{t-1}) \oplus (p^t - m_{t-1} - M_{t-1}) = () \oplus () \oplus () \oplus (p^t) = (p^t)$$

and $c_t = r_t \cdot \alpha_t = r_t \cdot (p^t)$. Alternatively, if $t > 0$ and there is an overrun at t, then by Theorem 1 again

$$\alpha_i = (p^t - m_{t-1} - M_{t-1}) \oplus c_{t-1} \oplus (M_{t-1} - m_{t-1}) \oplus rev(c_{t-1}) = (p^t) \oplus () \oplus () \oplus () = (p^t)$$

and

$$\begin{aligned} c_t &= (2rp^t - p^{t_1}) \oplus (p - 1 - r_i) \cdot \alpha_i \oplus (p^t - m_{t-1} - M_{t-1}) \oplus c_{t-1} \\ &= (2rp^t - p^{t_1}) \oplus (p - 1 - r_i) \cdot (p^t) \oplus (p^t) \oplus () \\ &= (2rp^t - p^{t_1}) \oplus (p - r_i) \cdot (p^t). \end{aligned}$$

Thus $n_k = p^t$ when $t > 0$.

By Equation (1),

$$\mu_k = 2r - 2\sum_{i=1}^{k-1} n_i - n_k,$$

which in turn equals $2r - 2\sum_{i=1}^{k} n_i + n_k = 2r - 2r + n_k = n_k$. Thus $\mu_k = n_k = p^t$. □

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