

The Mathematics of Poker

A Senior Comprehensive Project

by

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I hereby recognize and pledge to fulfill my responsibilities, as defined in the
Honor Code, and to maintain the integrity of both myself and the College
community as a whole.

Pledge:

name

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Abstract

This senior project will discuss the game of poker using different decks of cards with and without the addition of wild cards. We also solve a problem similar to one posed to Marilyn Vos Savant by Martin Gardner and apply a poker-like test to the table of random digits in the Mathematical Statistics textbook of John E. Freund.

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1 Introduction

The game of poker is a mathematical card game involving a deck of cards and a group of players. There are several different variations, styles of poker, and even different decks that can be played. In this case we will focus on the most basic form of poker, five card stud and use both the standard 52-card deck and 40-card deck. In five card stud each player is dealt five cards which can be called a hand. The goal for each player is to have the highest ranking hand of all those playing. In the standard game of poker we have a hierarchy of hands that are mathematically consistent. When a hand is ranked higher, it must occur less frequently, and if a hand is ranked lower, then it must occur more frequently. We will include the probabilities of each hand with two different decks and with the Aces being just low as well as high and low.

1.1 52-Card Deck with Aces Low

A standard 52-card deck consists of four different suits, hearts, spades, diamonds, and clubs, with each suit containing 13 different denominations, Ace, 2 through 10, Jack, Queen, King ranked from lowest to highest respectively. There are eight possible hands a player can achieve and each has a different probability than the rest. The total number of possible poker hands is $\binom{52}{5} = 2,598,960$. The number of possible hands containing:

One Pair: $\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1} = 1,098,240$

Two Pair: $\binom{13}{2} \binom{4}{2} \binom{4}{2} \binom{44}{1} = 123,552$

Three of a Kind: $\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} = 54,912$

Straight: $9 \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} = 9,216$

Flush: $\binom{4}{1} \binom{13}{5} = 5,148,$

Full House: $\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2} = 3,744$

Four of a Kind: $\binom{13}{1} \binom{4}{4} \binom{48}{1} = 624$

Straight Flush: $\binom{9}{1} \binom{4}{1} = 36.$

The probabilities of obtaining these hands are therefore:

One Pair: $\frac{1,098,240}{2,598,960} \approx .4224$

Two Pair: $\frac{123,552}{2,598,960} \approx .0475$

Three of a Kind: $\frac{54,912}{2,598,960} \approx .0211$

Straight: $\frac{9,216}{2,598,960} \approx .0035$

Flush: $\frac{5,148}{2,598,960} \approx .0020$

Full House: $\frac{3,744}{2,598,960} \approx .0014$

Four of a Kind: $\frac{624}{2,598,960} \approx .0002$

Straight Flush: $\frac{36}{2,598,960} \approx .0000139.$

1.2 52-Card Deck with Aces High and Low

Above we found the possible number of hands for each ranking with the Aces being the lowest card. In typical games the Aces are played as either the highest or the lowest ranking card. This adjustment will only effect the probabilities of a straight and a straight flush. The number of possible hands containing:

Straight: $10 \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} = 10,240$

Straight Flush: $\binom{10}{1} \binom{4}{1} = 40$.

The probabilities of obtaining these hands are therefore:

Straight: $\frac{10,240}{2,598,960} \approx .00394$

Straight Flush: $\frac{40}{2,598,960} \approx .000015$.

1.3 40-Card Deck with Aces Low

The 40-card deck consists of the same four suits. However, the denominations are Ace, 2 through 7, Queen, Jack, King from lowest to highest respectively. The forty card deck is in use in some European countries, like Italy, and in those countries the Jack outranks the Queen. With less cards in the 40-card deck we can assume that this will have a large impact on the statistics of each of the poker hands and possibly the hierarchy of hands as well. The total number of poker hands with a 40-card deck is $\binom{40}{5} = 658,008$.

The number of possible hands containing:

One Pair: $\binom{10}{1} \binom{4}{2} \binom{9}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1} = 322,560$

Two Pair: $\binom{10}{2} \binom{4}{2} \binom{4}{2} \binom{32}{1} = 51,840$

Three of a Kind: $\binom{10}{1} \binom{4}{3} \binom{9}{2} \binom{4}{1} \binom{4}{1} = 23,040$

Straight: $6 \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} = 6,144$

Flush: $\binom{4}{1} \binom{10}{5} = 1,008$

Full House: $\binom{10}{1} \binom{4}{3} \binom{9}{1} \binom{4}{2} = 2,160$

Four of a Kind: $\binom{10}{1} \binom{4}{4} \binom{36}{1} = 360$

Straight Flush: $\binom{6}{1} \binom{4}{1} = 24$.

In this case we can see that unlike the case with the 52-card deck a full house occurs more often than a flush. Therefore, for the hierarchy to be mathematically consistent we must rearrange the rankings so that a flush is more valuable than a full house. The probabilities of each hand are as follows:

One Pair: $\frac{322,560}{658,008} \approx .49$

Two Pair: $\frac{51,840}{658,008} \approx .0788$

Three of a Kind: $\frac{23,040}{658,008} \approx .035$

Straight: $\frac{6,144}{658,008} \approx .00934$

Flush: $\frac{1,008}{658,008} \approx .0015$

Full House: $\frac{2,160}{658,008} \approx .00328$

Four of a Kind: $\frac{360}{658,008} \approx .000547$

Straight Flush: $\frac{24}{658,008} \approx .000036.$

1.4 40-Card Deck with Aces High and Low

Similar to the case of the 52-card deck, when we allow the Aces to be high and low it will effect only the number of straights and straight flushes.

The number of possible hands containing:

Straight: $7 \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} = 7,168$

Straight Flush: $\binom{7}{1} \binom{4}{1} = 28.$

The probabilities of these two hands are as follows:

Straight: $\frac{7,168}{658,008} \approx .0109$

Straight Flush: $\frac{28}{658,008} \approx .000043.$

2 Wild Cards

Next, we will introduce Wild Cards into our 5-card stud game of poker and calculate the number of possible hands and probabilities as well. We will add Jokers into the deck, and they will act as our wild cards. These wild cards, when contained in the hand of a player, may act as any card in the deck. We will calculate the number of possible outcomes for each hand and their probabilities containing no wild cards and one wild card for both the 52 and 40 card decks. One important assumption must be made when we are calculating the hands: a hand containing a Wild Card(s) will obtain the highest ranking possible with that wild card(s). For example, if my hand contains two 3's, one 8, one Jack, and a Joker, this hand would be considered three of a kind since the Joker would become a 3. In this case, the Joker would not be an 8 or a Jack since two pair would be a lower ranked hand than three of a kind. Notice, the total number of possible hands when we add two jokers will be $\binom{54}{5} = 3,162,510$ and $\binom{42}{5} = 850,668$.

2.1 52-card Deck

2.1.1 52-card Deck with Aces Low, 1 Wild Card

We will look at each hand when the 52-card deck contains just a single Joker and obtain the number of outcomes and probabilities. When adding a joker to the 52-card deck a new hand is created; five of a kind is possible now. When we add one Joker to the deck we have $\binom{53}{5} = 2,869,685$ hands

possible. The number of possible outcomes of each hand with one Wild Card in the deck will be,

$$\mathbf{1 \text{ pair:}} \quad \binom{1}{0} \binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1} + \binom{1}{1} \binom{13}{4} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} = 1,281,280$$

$$\mathbf{2 \text{ pair:}} \quad \binom{1}{0} \binom{13}{2} \binom{4}{2} \binom{4}{2} \binom{44}{1} + 0 = 123,552$$

$$\mathbf{3 \text{ of a kind:}} \quad \binom{1}{0} \binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} + \binom{1}{1} \binom{13}{1} \binom{4}{2} \binom{12}{2} \binom{4}{1} \binom{4}{1} = 137,280$$

$$\mathbf{Straight:} \quad 9 \binom{1}{0} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} + 9(4^4) + 9(4^4) + 10(4^4) + 9(4^4) = 18,688$$

$$\mathbf{Flush:} \quad \binom{1}{0} \binom{4}{1} \binom{13}{5} + \binom{1}{1} \binom{4}{1} \binom{13}{4} = 8,008$$

$$\mathbf{Full House:} \quad \binom{1}{0} \binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2} + \binom{1}{1} \binom{13}{2} \binom{4}{2} \binom{4}{2} = 6,552$$

$$\mathbf{4 \text{ of a kind:}} \quad \binom{1}{0} \binom{13}{1} \binom{4}{4} \binom{48}{1} + \binom{1}{1} \binom{13}{1} \binom{4}{3} \binom{48}{1} = 3,120$$

$$\mathbf{Straight Flush:} \quad \binom{1}{0} \binom{9}{1} \binom{4}{1} + 37 \binom{1}{1} \binom{4}{1} = 184$$

$$\mathbf{5 \text{ of a kind:}} \quad 0 + \binom{1}{1} \binom{13}{1} \binom{4}{4} = 13.$$

The probabilities associated with the number of outcomes of each hand containing a single Joker are as follows:

$$\mathbf{1 \text{ pair:}} \quad \frac{1,281,280}{2,869,685} \approx .4465$$

$$\mathbf{2 \text{ pair:}} \quad \frac{123,552}{2,869,685} \approx .0431$$

$$\mathbf{3 \text{ of a kind:}} \quad \frac{137,280}{2,869,685} \approx .0478$$

$$\mathbf{Straight:} \quad \frac{18,688}{2,869,685} \approx .0072$$

$$\mathbf{Flush:} \quad \frac{8,008}{2,869,685} \approx .00279$$

$$\mathbf{Full House:} \quad \frac{6,552}{2,869,685} \approx .0023$$

$$\mathbf{4 \text{ of a kind:}} \quad \frac{3,120}{2,869,685} \approx .00109$$

$$\mathbf{Straight Flush:} \quad \frac{184}{2,869,685} \approx .000064$$

$$\mathbf{5 \text{ of a kind:}} \quad \frac{13}{2,869,685} \approx .0000045.$$

2.1.2 52-card Deck, Aces High and Low, 1 Wild Card

As before, when the Aces are allowed to be high and low our outcomes and probabilities will be different only in the case of the Straight Flush and Straight. The outcomes and probabilities are as follows:

Straight: $10 \binom{1}{0} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} + 10(4^4) + 10(4^4) + 11(4^4) + 10(4^4) = 20,736$

Straight Flush: $\binom{1}{0} \binom{10}{1} \binom{4}{1} + (10 + 10 + 11 + 10) \binom{1}{1} \binom{4}{1} = 204.$

The probabilities associated will be:

Straight: $\frac{20,736}{2,869,685} \approx .00723$

Straight Flush: $\frac{204}{2,869,685} \approx .000071.$

2.2 40-card Deck

2.2.1 40-card Deck with Aces Low, 1 Wild Card

Looking at the 40-card deck we will have different combinations leading to different outcomes and probabilities. When the Joker is added to the deck we have $\binom{41}{5} = 749,398$ poker hands possible. The number of outcomes for each hand are as follows:

1 pair: $\binom{1}{0} \binom{10}{1} \binom{4}{2} \binom{9}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1} + \binom{1}{1} \binom{10}{4} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} = 376,320$

2 pair: $\binom{1}{0} \binom{10}{2} \binom{4}{2} \binom{4}{2} \binom{32}{1} + 0 = 51,840$

3 of a kind: $\binom{1}{0} \binom{10}{1} \binom{4}{3} \binom{9}{2} \binom{4}{1} \binom{4}{1} + \binom{1}{1} \binom{10}{1} \binom{4}{2} \binom{9}{2} \binom{4}{1} \binom{4}{1} = 57,600$

Straight: $6 \binom{1}{0} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} + 6(4^4) + 6(4^4) + 7(4^4) + 6(4^4) = 12,544$

Flush: $\binom{1}{0} \binom{4}{1} \binom{10}{5} + \binom{1}{1} \binom{4}{1} \binom{10}{4} = 1,848$

Full House: $\binom{1}{0} \binom{10}{1} \binom{4}{3} \binom{9}{1} \binom{4}{2} + \binom{1}{1} \binom{10}{2} \binom{4}{2} \binom{4}{2} = 3,780$

4 of a kind: $\binom{1}{0} \binom{10}{1} \binom{4}{4} \binom{36}{1} + \binom{1}{1} \binom{10}{1} \binom{4}{3} \binom{36}{1} = 1,800$

Straight Flush: $\binom{1}{0} \binom{6}{1} \binom{4}{1} + (6 + 6 + 7 + 6) \binom{1}{1} \binom{4}{1} = 124$

5 of a kind: $0 + \binom{1}{1} \binom{10}{1} \binom{4}{4} = 10.$

The probabilities associated with the number of outcomes of each hand with a Joker added to the 40-card deck are as follows:

1 pair: $\frac{376,320}{749,398} \approx .5022$

2 pair: $\frac{51,840}{749,398} \approx .0692$

3 of a kind: $\frac{57,600}{749,398} \approx .0769$

Straight: $\frac{12,544}{749,398} \approx .0167$

Flush: $\frac{1,848}{749,398} \approx .00247$

Full House: $\frac{3,780}{749,398} \approx .0050$

4 of a kind: $\frac{1,800}{749,398} \approx .0024$

Straight Flush: $\frac{124}{749,398} \approx .0001655$

5 of a kind: $\frac{10}{749,398} \approx .000013.$

2.2.2 40-card Deck, Aces High and Low, 1 Wild Card

As before, when the Ace is allowed to be high and low our outcome and probabilities will be different only in the case of the Straight Flush and Straight. The outcomes and probabilities are as follows:

Straight: $7 \binom{1}{0} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} \binom{4}{1} + 7(4^4) + 7(4^4) + 8(4^4) + 7(4^4) = 14,592$

Straight Flush: $\binom{1}{0} \binom{7}{1} \binom{4}{1} + (7 + 7 + 8 + 7) \binom{1}{1} \binom{4}{1} = 144.$

The probability of these hands are as follows:

Straight: $\frac{14,592}{749,398} \approx .01947$

Straight Flush: $\frac{144}{749,398} \approx .000192$.

3 Conditional Probabilities

Definition: The conditional probability of an event B given an event A, denoted $P(B|A)$ ¹, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}. \quad (1)$$

3.1 “I have an Ace”

In this and the following section we consider and solve a problem suggested by one posed in a famous magazine column with Martin Gardner² and Marilyn vos Savant³. They considered the problem for Bridge hands and we solve it for Poker hands. Professor Lo Bello and I are playing five card draw poker with a standard 52-card deck. Professor Lo Bello says, “I have an Ace in my hand.” Let’s find the probability that Dr. Lo Bello has more than one Ace in his hand given he has an Ace in his hand already. So, our two events are **Event A:** Dr. Lo Bello has an Ace in his hand and **Event B:** Dr. Lo Bello has more than one Ace in his hand. Using formula (1) we need to find the probability of Event A as well as the probability of the intersection of Event A and B. The probability of Dr. Lo Bello having an Ace in his hand

¹Definition, pages 50-54 from Freund, Mathematical Statistics, second edition.

²Gardner’s column in “Scientific American,” pages 180-182.

³Savant’s “Ask Marilyn,” from Parade Magazine, September 8, 1991.

is simply 1 minus the probability that he has no Aces. Observe,

$$P(A) = 1 - \frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}} = \frac{18,472}{54,145} \approx .3412. \quad (2)$$

Next, we need to find the intersection of Event A and B, or the probability that Dr. Lo Bello has one Ace and more than one Ace in his hand. The probability will be 1 minus the probability of having no aces or only one ace in his hand. The probability of,

$$P(A \cap B) = 1 - \frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}} - \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} = \frac{2,257}{54,145} \approx .0417. \quad (3)$$

Then, still using (1) we will substitute (2) and (3) into our equation,

$$\frac{P(A \cap B)}{P(A)} = \frac{\frac{2,257}{54,145}}{\frac{18,472}{54,145}} = \frac{2,257}{18,472}.$$

Thus, the probability Dr. Lo Bello has more than one Ace in his hand given that he has an Ace in his hand is approximately .1222.

Let's do the same problem with the 40-card deck instead of the 52-card deck. So, we must find the probability that Professor Lo Bello has more than one Ace given he has one ace in his hand. Observe, the probability Dr. Lo Bello has an Ace is one minus the probability that he has no Aces,

$$P(A) = 1 - \frac{\binom{4}{0}\binom{36}{5}}{\binom{40}{5}} = \frac{3903}{9139} \approx .4271.$$

Similarly, we now need to find $P(A \cap B)$ which is one minus the probability Dr. Lo Bello has one or no Aces. Observe,

$$P(A \cap B) = 1 - \frac{\binom{4}{0}\binom{36}{5}}{\binom{40}{5}} - \frac{\binom{4}{1}\binom{36}{4}}{\binom{40}{5}} = \frac{97}{1,406} \approx .0689.$$

Using the $P(A)$ and $P(A \cap B)$ we obtain,

$$\frac{\frac{97}{1,406}}{\frac{3,903}{9,139}} = \frac{1,261}{7,806}.$$

Finally, the probability of $P(B|A) \approx .1615$.

3.2 “I have the Ace of Spades”

We now compare the probabilities calculated in the previous section with those obtained when we change the condition from “I have an ace” to “I have the ace of spades.” Similar to the first problem, Dr. Lo Bello and I are playing 5-card draw poker with a standard 52-card deck. However, this time Dr. Lo Bello says “I have the Ace of Spades in my hand.” What is the probability that Dr. Lo Bello has more than one Ace in his hand? So, **Event A**: Dr. Lo Bello has the Ace of Spades and **Event B**: Dr. Lo Bello has more than one Ace. Using (1) we will find the probability that Dr. Lo Bello has the Ace of Spades first and then find the intersection of events A

and B. Observe,

$$P(A) = \frac{\binom{1}{1}\binom{51}{4}}{\binom{52}{5}} = \frac{5}{52} \approx .0962, \quad (4)$$

The intersection of A and B is equal to the probability Dr. Lo Bello has the Ace of Spades minus the probability that Dr. Lo Bello has only the Ace of spades and no other aces. Let's find the intersection,

$$P(A \cap B) = \frac{5}{52} - \frac{\binom{1}{1}\binom{3}{0}\binom{48}{4}}{\binom{52}{5}} = \frac{461}{21,658} \approx .0213. \quad (5)$$

Now, we can use (1) and substitute what we found in (4) and (5) to find the conditional probability. Substituting we obtain,

$$\frac{P(A \cap B)}{P(A)} = \frac{\frac{461}{21,658}}{\frac{5}{52}} = \frac{922}{4,165}.$$

Therefore, the probability that Dr. Lo Bello has more than one Ace given he has the Ace of Spades is approximately .2214.

Now let's find the same probability with the 40-card deck this time. The events are the same as above so we must find the probability that Dr. Lo Bello has the Ace of Spades in his hand. Observe,

$$P(A) = \frac{\binom{1}{1}\binom{39}{4}}{\binom{40}{5}} = \frac{1}{8} = .125.$$

Now, we need to find the probability that Dr. Lo Bello has the Ace of Spades

and has more than one Ace. This will be the $P(A)$ minus the probability that Dr. Lo Bello has no additional Aces in his hand. We will now find the intersection,

$$P(A \cap B) = \frac{1}{8} - \frac{\binom{1}{1} \binom{3}{0} \binom{36}{4}}{\binom{40}{5}} = \frac{1,297}{36,556} \approx .0355.$$

Now, we will substitute the probabilities back into (1) to obtain,

$$P(B|A) = \frac{\frac{1,297}{36,556}}{\frac{1}{8}} = \frac{2,594}{9,139}.$$

The probability Dr. Lo Bello has more than one Ace given he has the Ace of Spades is approximately .2838.

Our evaluation therefore is that whether he plays with a 52-card deck or 40-card deck, Professor Lo Bello's probability of having an additional ace is greater when he reveals he has the Ace of Spades than when he just reveals he has an ace.

4 Random Number Table: Poker Test

Definition⁴: The moment generating function of the random variable $\mathbf{X}$ is $M_X(t) = E(e^{Xt})$.

Definition: $\mathbf{X}$ has Binomial Distribution with parameters n and θ if its probability function is

$$\beta(x : n, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}, \quad x = 0, 1, 2, \dots, n; n = 1, 2, \dots; 0 \leq \theta \leq 1.$$

Definition: $\mathbf{X}$ has Normal Distribution with parameter μ and σ if its probability density function is

$$f(x : \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; -\infty < x < \infty; -\infty < \mu < \infty; \sigma > 0.$$

Definition: $\mathbf{X}$ has Chi-square Distribution with ν degrees of freedom if its probability density function is

$$f(x : \nu) = \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}, \quad x > 0; \nu = 1, 2, 3, \dots$$

Theorem: If $\mathbf{X}$ has binomial distribution with parameter n and θ , then the mean of $\mathbf{X}$ is $n\theta$ and the variance of $\mathbf{X}$ is $n\theta(1 - \theta)$.

Theorem: If has normal distribution with parameters μ and σ , then the mean of $\mathbf{X}$ is μ and the variance of $\mathbf{X}$ is σ^2 .

Theorem: If $\mathbf{X}$ and $X_1, X_2, X_3, \dots$ are random variables such that

⁴Definitions and Theorems, from Freund, Mathematical Statistics, second edition.

$\lim_{x \rightarrow \infty} M_{X_n}(t) = M(t)$, then for large n , $P(a < X_n < b)$ may be well approximated by $P(a < X < b)$.

Theorem: If $\mathbf{X}$ has binomial distribution with parameters n and θ and Z has the standard normal distribution, then

$$\lim_{n \rightarrow \infty} M_{\frac{x - n\theta}{\sqrt{n\theta(1-\theta)}}} = M_Z(t).$$

Normal approximation to the binomial distribution: as a result of the previous two theorems it is common to approximate $P(a < \frac{x - n\theta}{\sqrt{n\theta(1-\theta)}} < b)$ by $P(a < Z < b)$ for any real numbers a and b such that $a < b$.

In this next section we will be testing a table of random digits for its randomness. There are several different tests to check these sets of random digits for randomness. Here we will be considering a poker test for randomness. It is important to test these sets of random digits because many simulations and experiments rely on the set of numbers and if the numbers were not random then the simulations and experiments would lose validity. We will be observing several five digit sequences to see whether they meet a specific condition, in this case are they representative of poker hands. For example, a random sequence such as **13843** would be considered one pair since there are two 3's and three other different numbers. We may consider the hand a head if it satisfies a given condition ("the hand is one pair") and a tail otherwise. We will run a chi-square analysis with our data for each hand individually to determine if our set of random numbers truly is

random. Suppose we toss a coin of parameter θ a total of n times. Let $\mathbf{X}$ be the random variable whose value x is the number of heads. Observe the following 2x2 contingency table,

	Observed	Expected
Heads	x	$n\theta$
Tails	$n - x$	$n(1 - \theta)$

If n is large, then the binomial random variable $\mathbf{X}$ will have approximately normal distribution with mean $n\theta$ and variance $n(1 - \theta)$. Notice that

$$\begin{aligned}
\frac{(x - n\theta)^2}{n\theta} + \frac{[(n - x) - n(1 - \theta)]^2}{n(1 - \theta)} &= \frac{(x - n\theta)^2}{n\theta} + \frac{[n - x - n + n\theta]^2}{n(1 - \theta)} \\
&= \frac{(x - n\theta)^2}{n\theta} + \frac{(n\theta - x)^2}{n(1 - \theta)} \\
&= (x - n\theta)^2 \left[\frac{1}{n\theta} + \frac{1}{n(1 - \theta)} \right] \\
&= (x - n\theta)^2 \left[\frac{1 - \theta + \theta}{n\theta(1 - \theta)} \right] \\
&= \frac{(x - n\theta)^2}{n\theta(1 - \theta)} \\
&= \left[\frac{x - n\theta}{\sqrt{n\theta(1 - \theta)}} \right]^2.
\end{aligned}$$

Since the standardized random variable within the brackets has approximately standard normal distribution, its square will have approximately a chi-square distribution with one degree of freedom.

Below in Figure 1 notice there are 800 random five digit sequences. We will treat each five digit sequence as a poker hand. The table may therefore be

TABLE IX

SAMPLE PAGES OF RANDOM NUMBERS

17623	47441	27821	91845	01654	50375	23041	44848
45054	58410	92081	97624	73750	68343	40727	81203
73700	58730	06111	64486	64103	22132	22596	14305
38574	03805	06885	93353	88445	83544	23627	79176
04981	17531	97372	39558	94180	71108	19121	11558
45639	02485	43905	01823	11433	12220	36719	35435
98832	38188	24080	24519	61838	68801	49856	21739
66538	03619	60906	93370	59908	63103	36855	19127
77580	87772	86877	57085	41018	69556	06402	03436
67125	98175	00912	11246	85095	01581	92299	06166
83808	98997	71829	99430	59705	78103	66740	41743
56462	58166	97302	86828	75094	55208	77905	20705
52728	15101	72070	33706	78425	68672	79455	94334
44873	35302	04511	38088	80088	86918	20787	05691
67352	41526	23497	75440	83345	95889	39333	86027
35370	14915	15569	54045	62896	00342	66647	57096
06081	74957	87787	68849	96498	38270	90532	54307
58478	99297	43519	62410	30974	47335	04918	42974
21211	77229	74967	96038	57901	06163	99162	53285
98964	64425	33536	15079	88494	80833	47735	53996
81496	23996	56872	71401	34883	00045	89682	86664
71361	41989	92589	69785	24373	46438	28935	63903
36341	20326	37489	34626	16828	79262	23678	05509
34183	22856	18754	60422	32723	27646	92335	57138
98272	13969	12429	03093	01542	75066	73921	97188
75689	70722	88553	83300	00100	12787	74100	95536
64204	95212	31320	03783	82697	03389	19303	21646
16374	42305	56300	94227	28137	17549	22698	72955
93852	74363	30951	41367	02248	21570	37796	83789
48907	79840	34607	62668	56175	82515	23348	42207
42669	82391	20435	79306	80020	21622	67659	07878
81018	15125	48087	01250	20271	23094	48372	77621
31413	33756	15218	81976	38734	98944	02658	90698
77600	15175	67415	88801	48183	24263	49297	32923
42009	79616	45210	73186	46163	34158	03177	51696
02955	84348	46436	77911	45658	15024	66664	18730
93611	93346	71212	24405	71128	15524	56666	14763
11509	95853	02747	61889	19041	42899	49464	93965
63899	30932	90572	89971	32672	67306	93040	94527
73581	76780	03842	64009	15820	45310	04391	15521
49604	14809	12317	78062	82810	18981	62581	31642
81825	76822	87170	77235	74772	80840	05816	39023
60534	44842	16954	99466	52931	38199	85632	23761
41788	74409	76177	55319	95385	87644	07222	99751
73818	74454	02371	94693	76695	33451	57139	90612
81585	55439	98095	55578	83560	50374	04410	57272
21858	39489	39251	70450	28355	62002	85994	35807
99458	68184	62119	20229	84684	54861	41330	66808
03249	74135	43003	63132	21135	92001	43896	55887
59638	31226	89860	45191	24236	01536	43897	41294

TABLE IX (continued)

79154	30193	15271	93296	02946	96520	81881	56247
91727	06463	12248	57567	85697	62000	87957	07258
45642	71580	21558	66457	26734	68426	52067	23123
32502	34568	78777	35179	47529	32353	95941	72169
33308	75930	24865	17426	76603	99339	40571	41186
77727	78493	94580	70091	47526	26522	11045	83565
95701	11180	73936	17628	70100	85732	10741	92951
56537	06383	83182	76927	86919	50200	50889	08493
89803	93604	55735	66973	41614	30074	23403	03656
71078	31823	64316	95567	17930	26194	53836	53692
69177	59277	47629	83874	24649	31845	25736	75231
32408	16630	01242	63119	79999	34661	54308	59358
23292	90422	75540	15843	76801	49594	81002	30397
97309	96673	16389	51437	62567	08480	61873	63162
24680	51909	97230	58136	49723	15275	09399	11211
84913	67895	18804	17691	42658	70183	89417	57676
92889	18906	66500	50915	31924	80621	90921	14937
07855	38892	50990	96766	65080	35569	79392	14937
19294	72581	77377	04652	02906	38119	72407	71427
27341	02507	41858	08436	75153	86376	63852	60357
57058	04222	54488	12019	14192	49525	78844	13664
92195	42593	56488	35402	32059	11548	86264	74406
14722	10715	58795	42800	81716	80301	96704	57214
23733	93318	77730	87614	43315	50483	02950	09611
88062	21506	01750	71326	27610	10769	09921	46721
31250	41996	31680	41783	81782	04769	36716	82519
42359	01761	28842	71562	19975	48346	91029	78902
22532	81701	03425	28914	08356	78855	18769	52843
59849	02370	02784	13711	29708	17814	31556	68610
72981	96423	68791	91084	88014	27583	78167	25057
87644	57353	90349	16448	94491	19238	17396	10592
17586	65524	20162	04712	50957	05072	53948	07850
32889	19504	66500	28904	50915	31924	80621	17495
72563	15076	23780	52815	49631	93771	80200	84022
94406	63865	44336	27224	95683	58162	45516	39761
57735	31752	14403	29836	86017	20264	94618	85979
40671	92727	68626	81631	77339	64605	82583	85011
13729	51708	54104	81331	61714	57933	37342	26000
14960	88896	72784	82054	15232	48027	15832	62924
31556	80163	80203	90928	41447	34275	10779	83515
79255	69253	60254	01653	23244	43524	16382	36340
42693	78972	60322	90462	53460	83542	25224	70378
67410	12916	87933	78840	53442	16897	61578	05032
99084	48028	07184	41635	55548	19096	04130	23104
97129	70847	91864	08549	18185	69229	02340	63111
11918	90871	60965	23555	02372	45690	38995	23121
36705	51302	93147	29479	51715	35492	61371	87132
24519	14186	51153	78998	24717	16786	42786	86985
65231	14168	46135	27156	78022	32604	87259	93702
35319	03793	60344	95970	55995	05275	62405	43313
45551	46877	58631	82654	29192	86922	31908	42703

Figure 1: Table IX, pages 444-445 from Freund, Mathematical Statistics, second edition.

considered to present a random sample of size 800 from a binomial population with parameters $n = 800$ and θ equal to the probability of the hand fulfilling the given condition (i.e. being one pair). We will perform a chi-square goodness of fit test to decide if the 800 hands may be considered a random sample from a population with the given binomial distribution. Our null hypothesis is that they may be considered random at the level of significance $\alpha = .05$, and the alternative hypothesis is that they may not be. First we will find the probability of: one pair, two pair, three of a kind, full house,

and four of a kind. Observe,

One Pair: $\frac{5!}{2!} \binom{10}{1} \left(\frac{1}{10}\right)^2 \binom{9}{3} \left(\frac{1}{10}\right)^3 = \frac{63}{125} = .504,$

Two Pair: $\frac{5!}{2!2!} \binom{10}{2} \left(\frac{1}{10}\right)^4 \binom{8}{1} \left(\frac{1}{10}\right) = \frac{27}{250} = .108,$

Three of a kind: $\frac{5!}{3!} \binom{10}{1} \left(\frac{1}{10}\right)^3 \binom{9}{2} \left(\frac{1}{10}\right)^2 = \frac{9}{125} = .072,$

Full House: $\frac{5!}{3!2!} \binom{10}{1} \left(\frac{1}{10}\right)^3 \binom{9}{1} \left(\frac{1}{10}\right)^2 = \frac{9}{1000} = .009,$

Four of a kind: $\frac{5!}{4!} \binom{10}{1} \left(\frac{1}{10}\right)^4 \binom{9}{1} \left(\frac{1}{10}\right) = \frac{9}{2000} = .0045.$

Now that we have our probabilities, we use the observed number of each hand and the expected number of each hand to see if our χ^2 value tells us to reject our null hypothesis. To do so we must find the $\frac{(O-E)^2}{E}$ value for each hand, case by case. In other words, we need to find the $\frac{(O-E)^2}{E}$ value for One pair and the value for the remaining hands that are not One pair, a coin toss scenario. Then, we will do the same for Two pair and the remaining hands that are not two pair. When we find both $\frac{(O-E)^2}{E}$ values for each result we will add them together to get our χ^2 value, which will tell us to reject or not to reject our null hypothesis. In the table below we have the Observed number of each hand, the expected number of each hand, the $\frac{(O-E)^2}{E}$ value, and a χ^2 value.

	Observed	Expected	$\frac{(O-E)^2}{E}$	χ^2
One Pair	381	403.2	1.2223	
Not One Pair	419	396.8	1.2420	
				2.4643
Two Pair	74	86.4	1.7796	
Not Two Pair	726	713.6	.2155	
				1.9951
Three of a kind	63	57.6	.5063	
Not Three of a kind	737	742.4	.0393	
				.5456
Full House	7	7.2	.00556	
Not Full House	793	792.8	.00005	
				.0056

Notice, we omitted the four of a kind from our analysis because we did not get large enough observed and expected values for the goodness of fit test. In this analysis we will set $\alpha = .05$. We also need to calculate the degrees of freedom, which is equal to one less than the number of rows, in this case $\nu = 2 - 1$ or 1. To determine whether or not we should reject the null or not we need to find our χ^2 value from Table V which is $\chi^2_{.05,1} = 3.841$. Now, observing our table above if any of our χ^2 values are greater than 3.841 we will reject the null hypothesis meaning the difference of what was expected and what was observed is statistically significant. However, we notice that

the bold χ^2 values in the table are all less than 3.841. Thus, in each case we will not reject the null hypothesis. The two pages of random digits pass the “Poker Test”.

References

- [1] Freund, John E. Mathematical Statistics. 2nd ed., Prentice-Hall, 1971.
- [2] Gardner, Martin “Scientific American”, October, 1959, pages 180-182.
- [3] Savant, Marilyn V. “Ask Marilyn”, Parade Magazine, September 8, 1991.