

Space, Time, and Simultaneity: A Minkowski Perspective

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April 29, 2025

Submitted to the Department of Mathematics in partial fulfillment of the
requirements for the degree of Bachelor of Science.

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I hereby recognize and pledge to fulfill my responsibilities, as defined in the
Honor Code, and to maintain the integrity of both myself and the College
community as a whole.

Pledge:

name

Acknowledgements

My senior project not only demonstrates the conclusion of my undergraduate degree in mathematics, but also, the finale of my experience here at Allegheny College. As such, it is necessary to me to thank those that assisted me with my project, and those that provided me guidance and mentorship throughout my years of schooling:

Professor Carswell, thank you for being my project advisor, and for teaching me that I was capable of doing challenging things. There is truly not a better lesson to be learned.

Professor Werner, thank you for being my second reader, for always being patient with my learning, and demonstrating what educating should be.

Professor Leech, thank you for being my advisor, and for encouraging me throughout my mathematical journey.

Coaches Wade, Tara, Marques, and Mourer, thank you for always being in my corner, and being not only the best coaches I ever had, but the best mentors as well. I would learn more from you all than in the classroom!

Katya, thank you for all the late night study sessions, listening to me try to explain how a pole can fit in a barn (if you run fast enough), and for always reminding me that numbers can't hurt me!

To *the family I made along the way*, thank you for being my best friends. Your constant support means everything to me, and you all have made my journey brighter and filled with laughter.

Mom, Dad, Emily, and Michael, thank you for always being there for me. You've all pushed me to keep going, no matter how frustrated I was. The motivation to complete my degree was derived (that's a math pun) from you all. Whether it was driving me to pole vaulting practice, giving me leftovers after Thanksgiving, or standing in the rain and snow at my track meets, you've always been the Mickey (or should I say Mickies) to my Rocky. I love you all!

Abstract

The concept of Minkowski spacetime evolved our understanding of space and time. The following study explores the mathematical structure of special relativity, demonstrating how Lorentz transformations preserve spacetime intervals and ensure the invariance of the speed of light across all inertial frames. Key principles and phenomena such as time dilation and length contraction are viewed through both algebraic and geometric lenses. To supplement understanding, the *twin paradox* and *barn and pole paradox* are analyzed, demonstrating how contradictions are resolved via Minkowski spacetime. Through mathematical arguments, this study aims to provide foundational understanding of relativistic motion and its implications for modern physics.

Contents

1	Introduction to Special Relativity	1
2	Minkowski Spacetime and Events	3
2.1	Definition of Spacetime Coordinates and 4D Vectors	3
2.2	Inner Products and the Minkowski Metric	3
3	Lorentz Transformations	8
3.1	Definition	8
3.2	Coordinate Transformation	10
3.3	Time Dilation	10
3.4	Length Contraction	12
3.5	Simultaneity	14
4	Light Cones and Causality	16
4.1	Vector Classification	16
4.2	Structure	16
5	The Twin Paradox	18
5.1	Setup	18
5.2	Resolution	19
6	Barn and Pole Paradox	21
6.1	The Setup	21
6.2	Coordinates in the Barn Frame S	22
6.3	Coordinates in the Pole Frame S'	23
6.4	Resolution	24
7	Conclusion	25
	References	26

1 Introduction to Special Relativity

For over two centuries, Newtonian mechanics dominated the world of spacetime, depending on the concepts of absolute space and time as unchanging. This described the motions of planets, everyday objects, and particles. However, consider the *Michelson-Morley Experiment* [3]: Imagine a beam of light being split into two perpendicular paths due to the reflection of a semi-silvered mirror angled at 45° , with each beam traveling to distant mirrors before reflecting back. This results in the recombination of beams. The scientists who performed this experiment expected the motion of the Earth to create an interference pattern in the now combined beams. To their surprise, this was not shown, resulting in the idea that the speed of light is constant in any direction, unaffected by the motion of the earth or the observers. This experiment challenged the concept of absolute space and time, which laid the foundation for Einstein's special theory of relativity.

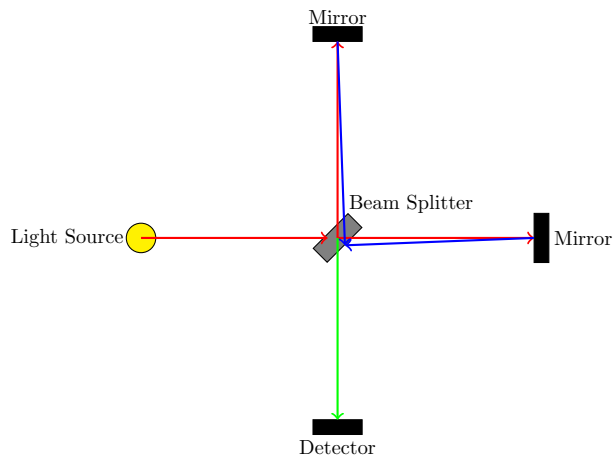


Figure 1: Diagram of the Michelson-Morley Experiment.

In 1905, Einstein introduced the special theory of relativity [1]. This was grounded on two paramount principles:

Principle of Relativity: The laws of physics are the same in all inertial frames of reference. This contradicts the idea of a universal ‘rest frame’.

Constancy of the Speed of Light: The speed of light in a vacuum, denoted as c , is always approximately 3×10^8 meters per second, regardless of the motion of the light source or the observer.

These two fundamental ideas obliterated Newtonian mechanics by demonstrating that space and time are interconnected and relative depending on the observer’s motion, resulting in the four-dimensional Minkowski Space-time.

2 Minkowski Spacetime and Events

2.1 Definition of Spacetime Coordinates and 4D Vectors

Minkowski spacetime M is a 4-dimensional real vector space with an inner product g , for which there exists a basis e_1, e_2, e_3, e_4 for M with the property that if $x = x^1e_1 + x^2e_2 + x^3e_3 + x^4e_4$ and $y = y^1e_1 + y^2e_2 + y^3e_3 + y^4e_4$, then

$$g(x, y) = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4 \quad (\text{See [2], pg. 2}). \quad (1)$$

It is assumed that the reader is familiar with the ideas of 4-dimensional real vector spaces, bases, and inner products. However, within this space, the inner product is indefinite. That is, the inner product can yield positive, negative, or zero values for all nonzero vectors. In the context of special relativity, elements of this space are called “*events*”, representing points in spacetime.

2.2 Inner Products and the Minkowski Metric

Within Minkowski spacetime, the Minkowski metric is utilized, and denoted as equation (1) (see [2], pg. 2)). This metric defines the Minkowski inner product $g(x, y)$, which encodes the geometric structure of spacetime. The Minkowski metric can also be denoted in matrix form:

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Applying the Minkowski metric to the difference between event vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$, the inner product produces a *spacetime interval*, denoted as s^2 , between them. That is,

$$s^2 = g(\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y}).$$

In component form, this becomes

$$s^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - (x_4 - y_4)^2.$$

A basis $\{e_1, e_2, e_3, e_4\}$ for M is said to be an orthonormal basis as it consists of mutually orthogonal unit vectors, satisfying

$$g(e_a, e_b) = \begin{cases} 1 & \text{if } a = b = 1, 2, 3, \\ -1 & \text{if } a = b = 4, \\ 0 & \text{if } a \neq b. \end{cases}$$

We now describe the conditions under which a linear transformation preserve the Minkowski inner product, as stated in the following proposition adapted from [2]. In what follows, the notation $\mathbf{x} \cdot \mathbf{y}$ refers to the Minkowski inner product defined previously in equation (1).

Proposition 2.1. *Let $L : M \rightarrow M$ be a linear transformation. Then the following are equivalent:*

- (a) *L is an orthogonal transformation, meaning that the Minkowski inner product is preserved. That is, for all vectors $\mathbf{x}, \mathbf{y} \in M$, $L\mathbf{x} \cdot L\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.*
- (b) *L preserves the quadratic form of M . That is, for all $\mathbf{x} \in M$, $L\mathbf{x} \cdot L\mathbf{x} = \mathbf{x} \cdot \mathbf{x}$.*
- (c) *L carries any orthonormal basis for M onto another orthonormal basis for M .*

Proof. Let $L : M \rightarrow M$ be a linear transformation.

We will start with showing that **(a) implies (c)**. Suppose L is an orthogonal transformation. So, $Lx \cdot Ly = x \cdot y$ for all $x, y \in M$. For an orthonormal basis $\{e_a\}$, we have the dot product $e_a \cdot e_b = \eta_{ab}$, where

$$\eta_{ab} = \begin{cases} 1, & \text{if } a = b = 1, 2, 3 \\ -1, & \text{if } a = b = 4 \\ 0, & \text{if } a \neq b \end{cases}.$$

So applying L , we obtain

$$Le_a \cdot Le_b = e_a \cdot e_b = \eta_{ab}$$

So, $\{Le_1, Le_2, Le_3, Le_4\}$ is also an orthonormal basis.

Now, we will show that **(c) implies (b)**. Suppose L maps an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ to another orthonormal basis $\{Le_1, Le_2, Le_3, Le_4\}$.

The original basis satisfies $e_a \cdot e_b = \eta_{ab}$. By the definition of orthonormal basis, the transformed basis must satisfy $Le_a \cdot Le_b = \eta_{ab}$. For any vector $\mathbf{x} \in M$, we can express it in terms of an orthonormal basis $\mathbf{x} = x^a e_a$, where repeated indices are summed according to the summation notation. So, we can write $L\mathbf{x} = x^a (Le_a)$. Taking the dot product, we compute $L\mathbf{x} \cdot L\mathbf{x} = x^a Le_a \cdot x^b Le_b$. Expanding this in summation notation, we obtain

$$L\mathbf{x} \cdot L\mathbf{x} = \sum_{a,b} x^a x^b (Le_a \cdot Le_b).$$

As L preserves the dot product of basis vectors, $Le_a \cdot Le_b = \eta_{ab}$. So,

$$L\mathbf{x} \cdot L\mathbf{x} = \sum_{a,b} x^a x^b \eta_{ab}.$$

Hence, $Lx \cdot Lx = x^a x^b \eta_{ab} = x \cdot x$, showing L preserves the quadratic form.

Finally, we show **(b) implies (a)**. Suppose L preserves the quadratic form of M . That is, for all vectors $\mathbf{x} \in M$, we have $L(\mathbf{x}) \cdot L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$. To show L preserves the Minkowski inner product, observe that for any vectors $\mathbf{x}, \mathbf{y} \in M$,

$$(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) - (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = 4(\mathbf{x} \cdot \mathbf{y}).$$

Since L preserves the quadratic form, we have

$$L(\mathbf{x} + \mathbf{y}) \cdot L(\mathbf{x} + \mathbf{y}) = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$$

and

$$L(\mathbf{x} - \mathbf{y}) \cdot L(\mathbf{x} - \mathbf{y}) = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}).$$

Subtracting, we obtain

$$L(\mathbf{x} + \mathbf{y}) \cdot L(\mathbf{x} + \mathbf{y}) - L(\mathbf{x} - \mathbf{y}) \cdot L(\mathbf{x} - \mathbf{y}) = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) - (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}).$$

Finally, simplifying, we get

$$4(L\mathbf{x} \cdot L\mathbf{y}) = 4(\mathbf{x} \cdot \mathbf{y}).$$

Thus, $L\mathbf{x} \cdot L\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$, showing L preserves the Minkowski inner product.

We have showed that all three given conditions are equivalent. $\square$

Transformations that satisfy these conditions will be examined further in Section 3, where we introduce the formal definition of Lorentz transformations.

3 Lorentz Transformations

3.1 Definition

As described in Section 2, we described the criteria needed for linear transformations that preserve the Minkowski inner product. We now focus on a specific transformation that demonstrates the symmetry of the space-time. A *Lorentz transformation* is a linear transformation $\Lambda : M \rightarrow M$ such that $g(\Lambda \mathbf{x}, \Lambda \mathbf{y}) = g(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in M$, where Λ is represented by a matrix. That is, a Lorentz transformation preserves the Minkowski inner product. In matrix notation, this condition is equivalent to

$$\Lambda^T \eta \Lambda = \eta. \tag{2}$$

This follows as Lorentz transformations preserve the Minkowski inner product (see [2], pg. 5).

This equation defines the set of Lorentz transformations as the group of 4×4 matrices that preserve the Minkowski metric. The matrix Λ maps events that occur in Minkowski spacetime to new coordinates while still preserving the very structure of spacetime. To connect this definition with coordinate computation, we restate the condition in terms of its component expressions following the formulation given in [2].

Proposition 3.1. *Let Λ_b^a be the components of a Lorentz transformation in Minkowski spacetime. That is, Λ is a linear transformation preserving the Minkowski inner product. Let η_{ab} be the components of the Minkowski metric, where η_{ab} are the standard values previously defined. Then, the following are equivalent:*

- $\sum_{a=1}^4 \sum_{b=1}^4 \Lambda_c^a \Lambda_d^b \eta_{ab} = \eta_{cd}$, for all $c, d = 1, 2, 3, 4$.
- $\Lambda^T \eta \Lambda = \eta$

Proof. Suppose $\Lambda^T \eta \Lambda = \eta$. Using this, consider the (c, d) component of both sides: $(\Lambda^T \eta \Lambda)_{cd} = \eta_{cd}$. Notice that $(\Lambda^T \eta)_{cb} = \sum_{a=1}^4 (\Lambda^T)_a^c \eta_{ab}$. So,

$$(\Lambda^T \eta)_{cb} \Lambda_d^b = \sum_{a=1}^4 \sum_{b=1}^4 (\Lambda^T)_a^c \eta_{ab} \Lambda_d^b.$$

Observe that $(\Lambda^T \eta)_{cb} \Lambda_d^b = (\Lambda^T \eta_{cb} \Lambda)_{cd} = \eta_{cd}$. Thus,

$$\sum_{a=1}^4 \sum_{b=1}^4 (\Lambda^T)_a^c \eta_{ab} \Lambda_d^b = \eta_{cd}.$$

Now, using the definition of transpose, we get

$$\sum_{a=1}^4 \sum_{b=1}^4 \Lambda_c^a \eta_{ab} \Lambda_d^b = \eta_{cd}.$$

This holds for all 16 values of (c, d) . Thus, the component equation is equivalent to $\Lambda^T \eta \Lambda = \eta$, showing equivalence. $\square$

Through this proof, it is demonstrated how Minkowski spacetime is preserved under transformations between coordinate systems moving at con-

stant velocity, or, an inertial frame, and reinforces the importance of Lorentz transformations within special relativity.

3.2 Coordinate Transformation

Let an inertial frame S observe an event occurring at (x, t) . Let S' be another inertial frame moving with constant velocity v in the x-direction relative to S . The coordinates of the same event in frame S' are denoted (x', t') , where x' denotes the position of the event in S' , and t' denotes the time of the event in S' . Also, $\beta = \frac{v}{c}$ denotes the velocity of S' , and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ denotes the Lorentz factor that scales time and space between the two frames. In this case, the Lorentz transformation reduces to

$$x' = \gamma(x - vt) \tag{3}$$

$$t' = \gamma\left(t - \frac{\beta x}{c}\right). \tag{4}$$

We now apply these formulas to derive time dilation and length contraction to describe how event coordinates transform between inertial frames in relative motion.

3.3 Time Dilation

Let S and S' be inertial frames of references with S' moving with constant velocity v in the x-direction relative to S . Suppose an event occurs within the same location as S' , but at two distinct times. As measuring the passage of time requires at least two distinct events- like how a clock ticks- we now must consider a pair of events occurring at the same position in the frame

of S' that is separated by time. That is, in the frame of S' , the events have coordinates (x', t'_1) and (x', t'_2) . As such, the elapsed time in S' is

$$\Delta t' = t'_2 - t'_1.$$

This interval represents proper time, denoted as $\Delta\tau$.

Now, we turn to how the proper time interval appears in the frame of S . Thus, we apply the inverse Lorentz transformation for time:

$$t = \gamma(t' + \frac{v}{c^2}x').$$

Applying this to both events and taking the difference, we get

$$\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x').$$

But, as the events occur within the same location in the frame of S' , $\Delta x' = 0$.

So, we have

$$\Delta t = \gamma\Delta t'.$$

As such, the time interval measured by S is longer than S' by a factor of γ . This phenomenon of moving clocks ticking slow is known as *time dilation*. We now formalize this in the following proposition from [2], in which time dilation is shown to be a direct consequence of Lorentz transformations.

Proposition 3.2. *Let two events be separated by a time interval $\Delta\bar{x}^4$ in an inertial frame where $\Delta\bar{x}^1 = \Delta\bar{x}^2 = \Delta\bar{x}^3 = 0$. Then, under a Lorentz*

transformation with relative velocity v , the transformed time interval satisfies $\Delta\bar{x}^4 = \gamma\Delta\bar{x}^4 = \frac{\Delta\bar{x}^4}{\sqrt{1-\beta^2}}$.

Proof. Suppose two events are separated by a time interval $\Delta\bar{x}_4$. Let $\Delta\bar{x}^1 = \Delta\bar{x}^2 = \Delta\bar{x}^3 = 0$. Thus, substituting $\Delta\bar{x}^1 = 0$ into (4), we get

$$\Delta\bar{x}^4 = \gamma(\Delta\bar{x}^4 - \frac{v(0)}{c^2}) = \gamma\Delta\bar{x}^4 = \frac{\Delta\bar{x}^4}{\sqrt{1-\beta^2}},$$

as desired. □

While the derivation above was performed through the frame of S , the scenario is symmetric as from the frame of S' , it is rather the clock in S that seemingly ticks differently. This mutual difference in time is a result of the symmetry of Lorentz transformations.

3.4 Length Contraction

Let S and S' be inertial frames of references, with S' moving with constant velocity v in the x-direction relative to S . Suppose a pole is at rest in S' with endpoints located at location x'_1 and x'_2 in that frame. The length of the pole in S' , denoted as L' , is given by

$$L' = x'_2 - x'_1.$$

Now, to determine how the length of the pole is observed in the frame of reference S , we must compute the spatial separation of the two endpoints at

the same instant in S . As such, we utilize the inverse Lorentz transformation

$$x = \gamma(x' + vt)$$

For this scenario, we consider the measurements of the pole to be taken at a fixed time in the frame of S . As the pole is moving in S , the position of its endpoints at time t are given by

$$x_1 = \gamma(x'_1 + vt), \quad x_2 = \gamma(x'_2 + vt).$$

Now, taking the difference, we get

$$x_2 - x_1 = \gamma(x'_2 - x'_1) = \gamma L'.$$

So, solving for L' , we find the length for the reference frame S to be

$$L = x_2 - x_1 = \gamma L'.$$

Thus, we get

$$L' = \frac{L}{\gamma}.$$

Therefore, the length of the pole as measured in the frame where it is moving is shorter than its length at rest by the Lorentz factor γ . This phenomenon is known as *length contraction*.

Similar to time dilation, length contraction is also symmetric. Consider how the frame of S' sees that the pole in S is contracted. Each observer

considers that the pole length is shorter along the direction of relative motion. Again, this mutual difference in length contraction is a result of the symmetry of Lorentz transformations.

3.5 Simultaneity

Let S and S' be inertial frames, with S' moving with constant velocity v in the x -direction relative to S . Suppose two events occur at distinct positions x_1 and x_2 in the frame of reference S , at times t_1 and t_2 , respectively. Now, suppose the two events are simultaneous in S such that $t_1 = t_2$. To determine if these events truly occur simultaneously in S' , we use the Lorentz transformation for time

$$t' = \gamma(t - \frac{v}{c^2}x).$$

Applying this to both events and taking the difference, we get

$$\Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x),$$

where $\Delta t = t_2 - t_1 = 0$. Applying this, we get

$$\Delta t' = -\gamma \frac{v}{c^2} \Delta x.$$

Supposing $\Delta x \neq 0$, it follows that $\Delta t' \neq 0$. Thus, two events that were simultaneous in one inertial frame may not be observed as simultaneous in another frame moving relative to the first. This phenomenon is known as *relativity of simultaneity*, which demonstrates that the ordering of events is

dependent upon an observer's inertial frame of reference.

To visualize this, consider the classic *train and platform thought experiment*. Let Observer A stand at the midpoint of a stationary platform (frame S), and Observer B ride at the midpoint of the train moving with constant velocity v . Now, consider a bolt of lightning that strikes both the front and back of the train simultaneously in the platform frame. As Observer A is equidistant from both ends of the train, the light from each bolt of lightning reaches them at the same time. This demonstrates simultaneity in S .

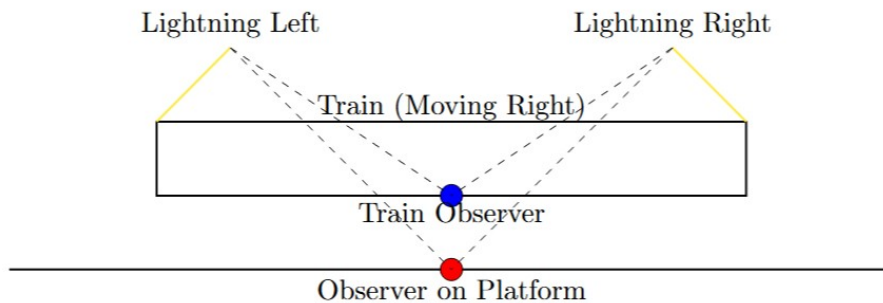


Figure 2: A visualization of the experiment.

Alternatively, from Observer B's perspective, they are moving towards the light from the bolt striking the front of the train, and away from the light from the bolt striking the back. As the speed of light is constant in all inertial frames, the front flash reaches Observer B first. Thus, the lightning must not be simultaneous in S' . This experiment illustrates that simultaneity is not absolute, but depends on the observer's frame of reference.

4 Light Cones and Causality

4.1 Vector Classification

Given a nonzero vector $\mathbf{x} \in M$, consider the sign of $g(\mathbf{x}, \mathbf{x})$, which represents the squared spacetime interval associated with $\mathbf{x}$. As stated in Section 2, the inner product is indefinite. Thus, this value may be positive, negative, or zero. This leads to the following classification of vectors:

- $\mathbf{x}$ is lightlike (null) if $g(\mathbf{x}, \mathbf{x}) = 0$.
- $\mathbf{x}$ is timelike if $g(\mathbf{x}, \mathbf{x}) < 0$.
- $\mathbf{x}$ is spacelike if $g(\mathbf{x}, \mathbf{x}) > 0$.

These classifications demonstrate the potential causal relationship between events in spacetime. Timelike vectors represent possible worldlines of objects moving incredibly fast, lightlike vectors describe the path of massless particles, and spacelike vectors connect causally disconnected events.

4.2 Structure

Given an event $O \in M$, the collection of all lightlike vectors originating from O forms a *lightcone*. The lightcone divides spacetime into three regions:

- The future lightcone contains all timelike and lightlike vectors that are future-directed.
- The past lightcone contains all timelike and lightlike vectors that are past-directed.

- The exterior of the lightcone contains all the spacelike vectors.

The lightcone structure determines the causal relationship between events. That is, events occurring within the past lightcone of O can influence O , while the events within the future lightcone of O can be influenced by O . Meanwhile, events outside the lightcone are causally disconnected from O . Only events that are timelike or lightlike separated can be related to it. Additionally, as the spacetime interval remains invariant under Lorentz transformations, the lightcone structure is preserved under all inertial frames. This allows the classification of vectors and causal relationships to be observed, regardless of an observer's inertial frame. Observe the figure below:

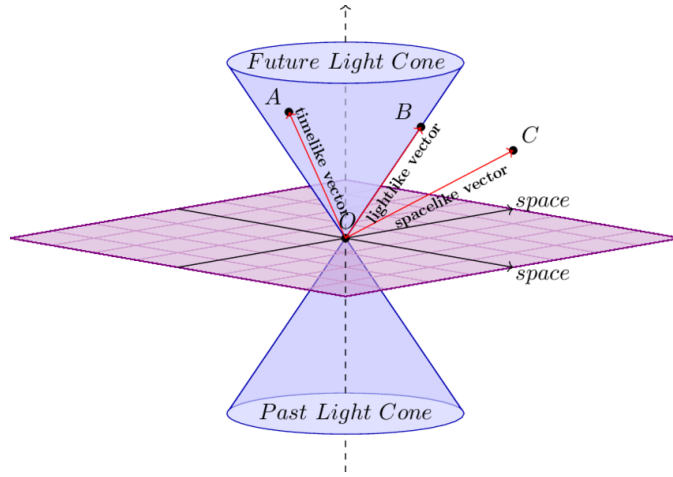


Figure 3: A lightcone in Minkowski spacetime (see [4]).

5 The Twin Paradox

The *twin paradox* demonstrates the asymmetry that arises when two observers follow different worldlines in Minkowski spacetime. This apparent paradox, first proposed by Einstein [1], can be resolved through the very structure of relativity.

5.1 Setup

Let Twin A remain at rest on Earth in an inertial frame S . Let Twin B depart Earth moving with constant velocity v heading towards a distant star. When Twin B arrives at the star, they turn around, and return at the same velocity. While Twin B is traveling, their motion is inertial, but the direction change at the star involves rapid de-acceleration and acceleration.

Let D denote the total distance the trip takes, and suppose each leg takes a coordinate time $\Delta t = \frac{D}{v}$ in Twin A's frame of reference. Within this frame, Twin A stays at rest. This results in a straight worldline, and that they experience coordinate time

$$\Delta t_A = 2\frac{D}{v}.$$

Alternatively, Twin B travels with constant velocity v . Thus, their proper time is subject to time dilation:

$$\Delta\tau = \frac{\Delta t}{\gamma}.$$

Thus, for Twin B's roundtrip, we get

$$\Delta\tau_B = 2\frac{\Delta t}{\gamma}, \quad \Delta t_A = 2\Delta t.$$

As $\gamma < 1$, it follows that

$$\Delta\tau_B < \Delta t_A.$$

This result demonstrates that Twin B experiences less proper time than Twin A.

5.2 Resolution

While in this paradox, each twin observes that the other's clock is ticking slower during inertial motion. But, the true asymmetry is due to Twin B switching frames at the star, breaking the symmetry between observers. In Minkowski spacetime, proper time relates to the length of an observer's worldline. A straight worldline demonstrates maximum proper time. So, Twin A, who remains at rest, experience the greatest proper time. In contrast, Twin B's worldline consists of two straight segments joined at a point. While time dilation applies to both segments, the point represents a shift in simultaneity. When this occurs, Twin B now sees that Twin A's clock has jumped forward.

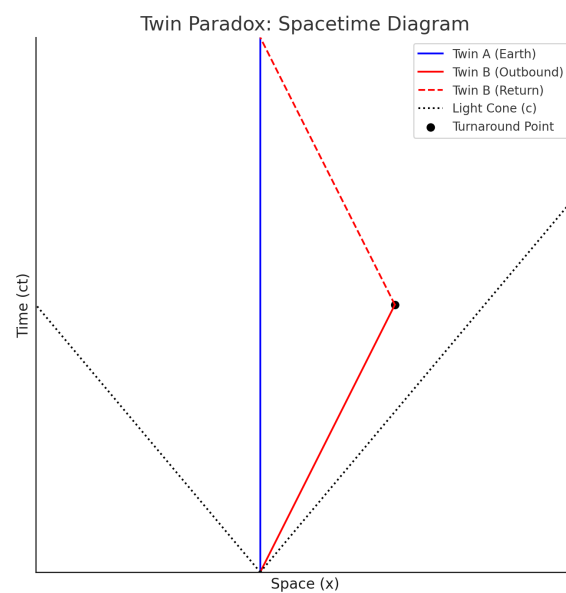


Figure 4: a spacetime diagram showing Twin A's straight worldline and Twin B's piecewise path.

6 Barn and Pole Paradox

The *Barn and Pole Paradox* presents that length contractions and the relativity of simultaneity could contradict one another. However, viewing this so called “paradox” through the lens of Lorentz transformations and the geometry of Minkowski spacetime, the apparent contradiction is resolved.

6.1 The Setup

Suppose there is a barn with a rest length of $L = 8m$ that is stationary in an inertial frame S . Let a runner carry a pole with a rest length of $L' = 16m$, moving with constant velocity v along the spatial x-axis. The runner’s rest frame of reference is denoted as S' . In this frame, the barn is moving toward the pole. We now define the following events for both frames of reference:

- **Event A** occurs when the front of the pole reaches the barn.
- **Event B** occurs when the front of the pole reaches the back of the barn.
- **Event C** occurs when the end of the pole reaches the front of the barn.

These events occur at specific coordinates in spacetime, which differ depending on the frame of reference. We now examine these coordinates in both frames.

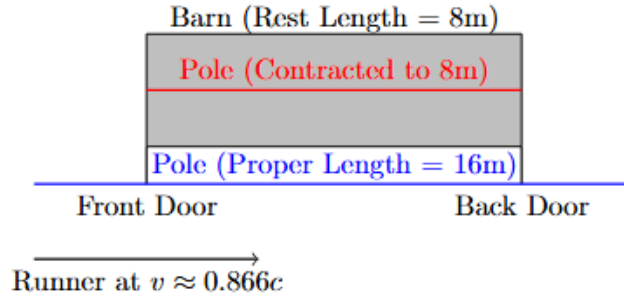


Figure 5: A visual aid of the Barn and Pole Paradox

6.2 Coordinates in the Barn Frame S .

Within the frame S , the barn is at rest and the pole is moving. We seek a velocity v such that the contracted length of the pole equals the barn length. Substituting $L = 8$ and $L' = 16$ into the length contraction formula, we find

$$8 = 16\sqrt{1 - \beta^2} \implies \beta = \frac{\sqrt{3}}{2}, \gamma = 2.$$

Let the front of the pole arrive at the front of the barn at time $t = 0$. The time for the front of the pole to reach the back of the barn is given by

$$t = \frac{d}{v} = \frac{8}{\frac{\sqrt{3}c}{2}} = \frac{16}{\sqrt{3}c}.$$

We can now show the events in the barn's frame:

- Event A occurs at $(x_A, t_A) = (0, 0)$.
- Event B occurs at $(x_B, t_B) = (8, \frac{16}{\sqrt{3}c})$.
- Event C occurs at $(x_C, t_C) = (0, \frac{16}{\sqrt{3}c})$.

Observe that $t_B = t_C$. This suggests Events B and C are simultaneous, resulting in the idea that the pole fits entirely in the barn.

6.3 Coordinates in the Pole Frame S' .

Within the frame S' , the pole is at rest and the barn moves toward it with velocity $v = \frac{\sqrt{3}c}{2}$. Due to length contraction, the barn's length in this frame is

$$L' = \frac{8}{\gamma} = \frac{8}{2} = 4.$$

We now can apply the Lorentz transformation for position and time for each event.

Event A occurs at $x_A = 0$, and $t_A = 0$. Substituting these into our transformation equations, we get

$$x'_A = 2(0 - v \cdot 0) = 0, \quad t'_A = 2(0 - 0) = 0.$$

Next, Event B occurs at $x_B = 8$, and $t_B = \frac{16}{\sqrt{3}c}$. Again, substituting, we get

$$x'_B = 2(8 - \frac{16\sqrt{3}c}{2}) = 2(8 - 8) = 0, \quad t'_B = 2(\frac{16}{\sqrt{3}c} - \frac{8\sqrt{3}c}{c^2}) = \frac{8}{\sqrt{3}c}.$$

Lastly, Event C occurs at $x_C = 0$, and $t_C = \frac{16}{\sqrt{3}c}$. Following the same process as above, we get

$$x'_C = 2(0 - (\frac{\sqrt{3}c}{2})(\frac{16}{\sqrt{3}c})) = -16, \quad t'_C = 2(\frac{16}{\sqrt{3}c} - \frac{0}{c^2}) = \frac{32}{\sqrt{3}c}.$$

We now summarize the events in the pole's frame:

- Event A occurs at $(x'_A, t'_A) = (0, 0)$.
- Event B occurs at $(x'_B, t'_B) = (0, \frac{8}{\sqrt{3}c})$.
- Event C occurs at $(x'_C, t'_C) = (-16, \frac{32}{\sqrt{3}c})$.

Notice that $t'_B \neq t'_C$, suggesting that events that were simultaneous in the barn's frame are not simultaneous in the pole's frame.

6.4 Resolution

In the barn's frame S , the pole is moving, and Events B and C occur simultaneously, suggesting that the pole fits entirely within the barn. Thus, both doors can close at the same time without contradiction. Meanwhile, in the pole's frame S' , the pole is at rest and the barn is moving, and Event B occurs prior to Event C. Thus, the entire pole is never inside the barn at a single moment, resulting in the barn doors not closing. These differing accounts of events do not represent a true paradox, but rather an example of the relativity of simultaneity. Each observer uses the same physical laws and transformations, which yields equally valid frames of references, and demonstrates the necessity of the four-dimensional structure of Minkowski spacetime to interpret motion, simultaneity, and causality.

7 Conclusion

The study of Minkowski spacetime replaced Newtonian absolutes with a relativistic framework, reshaping our understanding of space and time. Exploring Lorentz transformations provided a structure that preserves the spacetime interval and ensures the speed of light is invariant across all inertial frames. This aids in explaining the phenomena of causality, lightcones, and relativistic paradoxes.

The barn and pole paradox demonstrates how an event that appears to be simultaneous in one frame may not be simultaneous in another frame. The resolution of the paradox can be seen through Lorentz transformations. This solidifies the necessity of observing spacetime as a four-dimensional construct, so observer in differing inertial frames experience distinct, yet equally valid versions of events. Similarly, the twin paradox reinforces the interplay between acceleration, time dilation, and simultaneity.

This study focuses on the algebraic and geometric representations of Minkowski spacetime. The principles within this structure can be used to solve apparent paradoxes, and also, provide foundational understanding to topics in physics such as general relativity. Our understanding of spacetime will undoubtedly evolve. But, insights and understandings from Minkowski spacetime remain indispensable and pivotal in the shaping of physics.

References

- [1] Einstein, A. (1905). Zur Elektrodynamik bewegter Körper. *Annalen der Physik*, 17, 891–921.
- [2] Naber, G. L. (2012). *Spacetime and Singularities: An Introduction*. Cambridge University Press.
- [3] Resnick, R. (1968). *Introduction to Special Relativity*. John Wiley & Sons.
- [4] Grothus, M. (2023). A light cone in 2+1 Minkowski spacetime. ResearchGate. (Accessed: March 16, 2025).