A Dive Into the Ramsey Realm

A Senior Comprehensive Project
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April 19, 2025

Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of Bachelor of Science.

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I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

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name

Acknowledgements

Not only will this senior project mark the end of my undergraduate degree in mathematics, but also the conclusion of my academic experience at Allegheny College. It is of highest importance to thank those who have assisted me in composing this project, as well as those who have mentored and guided me throughout my entire academic career.

Professor Werner, thank you for being my project advisor and for encouraging me to try something new. You have taught me to be patient with challenging topics, and for that I will be forever thankful.

Professor Carswell, thank you for being my second reader and for supporting me throughout my mathematical journey at Allegheny.

Professor Lo Bello, thank your for being my advisor and for your continuous support in my life outside of academics. You have taught me to keep a positive attitude in everything I do, a lesson I will hold onto forever.

Coach Groff, Coach McCord, and Coach Sorensen, thank you for being the best trio of coaches I could have ever asked for. Your support for me and trust in me has allowed me to grow in ways I never could have imagined. You've allowed me to grow on the course, in the classroom, and in many other ways that I will be forever thankful for. The opportunities I have embraced because of you three have been both amazing and memorable, and I cannot thank you enough.

Friends and family, it would be hard to come by a stronger support system than you all. Through my ups and downs, each and every one of you has stuck by my side and supported me, something for which I will always

be thankful.

Mom, Dad, Leyton, and Josephine, your support for me throughout pursuing my degree has been second to none. Without you guys in my corner, anything that I have accomplished would not have been possible. I love you all.

Finally, to Allegheny. Four years ago, I made the decision to continue my academic and athletic career at this institution, and to this day it was the best decision of my life. Thank you for being a second home that has provided endless opportunities to not only me, but every student on campus. Thank you for all the connections, friendships, and relationships you have brought into my life. They will forever be cherished.

Abstract

This paper studies the fundamentals of Ramsey Theory, a branch of combinatorial mathematics that focuses on the appearance of order in large structures. We will work our way from studying the basics of Graph Theory to solving geometrical problems using Ramsey methods.

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1 Introduction

Ramsey Theory, named after British mathematician and philosopher Frank P. Ramsey, is a branch of mathematics that uses combinatorics and graph theory to find order in large sets of data. Similar to other theories in mathematics, Ramsey Theory has continuously evolved over time using prior results and ideas, such as David Hilbert's Cube Lemma of 1892 [10]. We should note that the symbol [m] represents the starting segment of positive integers:

$$[\mathrm{m}] = \{1, 2, ..., m\}.$$

The Hilbert Cube Lemma. *For every pair of positive integers r, n, there exists a least positive integer m = H(r, n) such that in every r-coloring of [m] there exists a monochromatic n-dimensional affine cube.*

While Hilbert's lemma was technically the first example of Ramseyan mathematics, it was appreciated by very few mathematicians. Instead, these mathematicians used Hilbert's work to develop other results that are more complex. Before diving into any of these, we first will go over some fundamental concepts of graph theory.

From there, we arrive at our party and observe a well-known problem in Ramsey theory. This sets us up to prove Ramsey's theorem and compute a few basic Ramsey numbers for two colors. We will see that not many Ramsey numbers have values that are know, for they are very complex by nature. Because of this, we must prove some upper and lower bounds for these numbers, giving us at least an idea of what their values may be. We

then study Ramsey numbers dealing with more than two colors. As we will see, very few of these numbers are known and it takes mathematical skills far beyond our level to compute them. It is interesting, however, to work with these numbers and eventually find one of their values.

Moving forward, we look at how Ramsey concepts can be used in a geometric way. Connecting different mathematical topics allows our brains to work in a different gear, allowing that moment of satisfaction when everything comes together. To close, we will look at a few examples using Ramsey methods and conclude our study.

2 Fundamentals of Graph Theory

Before we dive into the basics of Ramsey Theory, we must first understand some fundamental ideas of Graph Theory. So, let us first note some basic definitions [3].

Definition 2.1. A **graph** $G = (V(G), E(G))$ consists of two finite sets: $V(G)$, the vertex set of the graph, often denoted by just V, which is a nonempty set of elements called vertices, and $E(G)$, the edge set of the graph often denoted by just E, which is a possibly empty set of elements called edges, such that each edge e in E is an unordered pair of vertices u, v, called the end vertices of e.

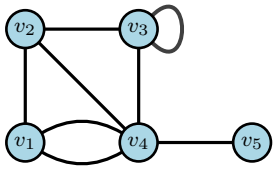


Figure 1: Example of a graph G.

Definition 2.2. A **loop** is an edge that connects any vertex to itself.

Definition 2.3. If two or more edges of a graph G have the same end vertices then these edges are called **parallel.**

Most of the graphs we will study in Ramsey Theory are considered to be **simple**, meaning they have no loops or parallel edges. Figure 1 is a example of a non-simple graph with a loop at vertex v_3 and the edges between v_1 and v_4 being parallel to each other. Figure 2, on the other hand, is an example of a simple graph.

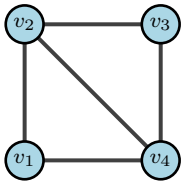


Figure 2: Example of simple graph.

For a graph G, we define the **degree** of a given vertex v_i to be the number of times v_i is an end vertex of an edge. We denote this degree of v_i as $d(v_i)$. For example, when looking at Figure 2, we see that $d(v_1) = d(v_3) = 2$, and $d(v_2) = d(v_4) = 3$.

Now that we have a few fundamental definitions down, let's focus our attention on a special type of graph.

Definition 2.4. Denoted as K_n, where n is the total number of vertices, a **complete** graph is a simple graph in which each pair of distinct vertices is joined by an edge [3].

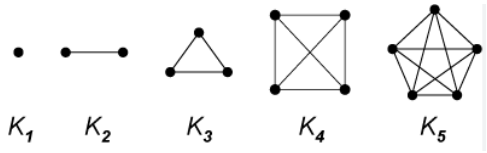


Figure 3: Complete graphs $K_1 - K_5$.

Complete graphs are used heavily in studying Ramsey Theory, so it is crucial we are able to understand what they are and how they work. Counting the number of edges of a given complete graph will be needed, which we can do using the **First Theorem of Graph Theory**.

Theorem 2.5 (First Theorem of Graph Theory [3])**.** *For any graph G with e edges and n vertices, $v_1, ..., v_n$,*

$$\sum_{i=1}^{n} d(v_i) = 2e.$$

Proof. The proof of this theorem is straightforward. In fact, since each edge has two end vertices, all edges are counted twice when the degrees of the vertices are summed. □

For example, consider the complete graph K_4:

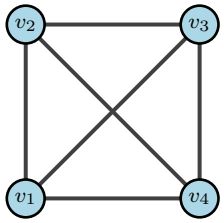


Figure 4: Complete graph with four vertices, K_4.

In Figure 4, we see that $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$; thus,

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) = 4(3) = 12 = 2(6).$$

Using Theorem 2.4, we arrive at the following corollary.

Corollary 2.6. *Any complete graph K_n has $\frac{n(n-1)}{2}$ edges.*

Proof. Take any complete graph K_n. We know that each vertex in K_n is connected to the other $n-1$ vertices. When multiplying n by $n-1$, we count each edge twice, once for each vertex it connects to. Thus, the quantity $\frac{n(n-1)}{2}$ is equal to the total number of edges in our complete graph K_n. □

Before studying any further, we must note the following definition.

Definition 2.7. Let H be a graph with vertex set $V(H)$ and edge set $E(H)$, and similarly, let G be a graph with vertex $V(G)$ and edge set $E(G)$. Then we say that H is a **subgraph** of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

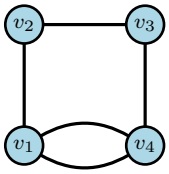


Figure 5: Subgraph of graph G in Figure 1.

In Ramsey Theory, we will be focusing on special types of subgraphs that are colored a certain way. So, let's consider the following definitions [3].

Definition 2.8. For any graph G, a complete subgraph of G is called a **clique** of G.

Definition 2.9. For any graph G, an **edge-coloring** of G is a function that assigns all elements of $E(G)$ a certain color.

Consider the following two examples of cliques:

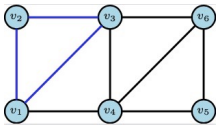


Figure 6: Blue K_3 clique

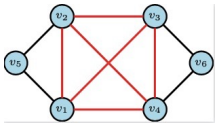


Figure 7: Red K_4 clique

Here, we see that our cliques are **monochromatic**, meaning all the edges of each clique are of the same color.

There is one final theorem we must prove before moving forward, as this theorem will be used throughout our research multiple times.

Theorem 2.10 (Pigeon-hole Principle)**.** *Let n and k be positive integers, and let $n > k$. Suppose we have to place n identical balls into k identical boxes. Then there will be at least one box in which we place at least two balls [2].*

Proof. We prove this theorem by assuming the contrary to be true, eventually arriving at a contradiction. So, let us assume there is no box with at least two balls. This means each of our k boxes has either 0 or 1 ball in it. Call the number of boxes that have zero balls in them m, meaning certainly that $m \geq 0$. Then, we have $k - m$ boxes that have one ball. However, this means the total number of balls placed into our k boxes is $k - m$. This is a contradiction, for our theorem states we have to place n balls in these boxes and $k - m \leq k < n$. Therefore, our assumption that there is no box with at least two balls must be false, and we are guaranteed at least one box with two balls in it. □

With the Pigeon-hole principle proven, we now have enough information on graph theory to move into the fundamentals of Ramsey theory. Instead of diving straight into material, we first consider a famous problem that happens to take place at a party.

3 The Party Problem

One of the most well-known problems in Ramsey Theory is the Party Problem. Proposed by Hungarian mathematician Paul Erdös, the question being asked is as follows: What is the smallest number of people needed in a room to guarantee that either three people know each other or three people do not know each other? The wording of this question may trick our minds into thinking the number of people is relatively high; however, using some graph theory, we can find the answer to be quite small.

To solve this, we will observe complete graphs of different sizes and color their edges blue or red. One color denotes that people know each other, while the other does the opposite. Ultimately, we are trying to guarantee ourselves a monochromatic triangle with this coloring. So, let us first observe the case with four people by coloring the edges of K_4:

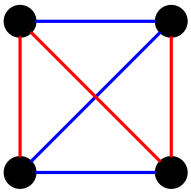


Figure 8: Coloring of complete graph with four people.

We see that the complete graph for four people does not guarantee us a monochromatic triangle. Therefore, four people is not enough to answer our question. The case of five people is very similar, and an example is as follows:

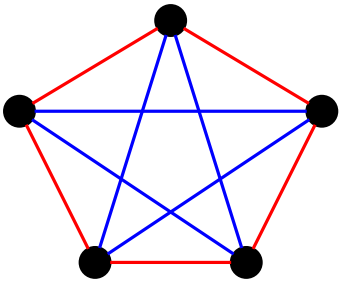


Figure 9: Coloring of complete graph with five people.

Again, we see that five people are not enough to answer our question. However, the case of six people is a little bit different. Observe the following two figures:

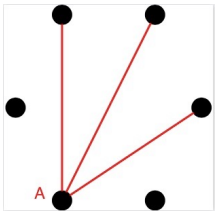


Figure 10: Relations from vertex A [1]

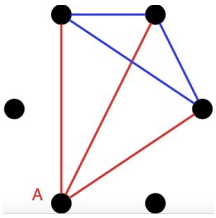


Figure 11: Monochromatic triangle [1]

We recognize that for any vertex in a complete graph K_6, there will be at least three adjacent edges of the same color. By adjacent, we mean edges that stem from the same vertex. Without loss of generality, consider vertex A in Figure 10 with three red edges. The vertices at the other ends of these edges are now our focus. When any of the edges connecting these vertices is colored red, we have our desired monochromatic red triangle. Now, consider when all of these connecting edges are colored blue. This results in Figure 11, where we also have our desired monochromatic blue triangle. Thus, in either case, six vertices are sufficient to guarantee a monochromatic triangle.

4 Ramsey's Theorem

Why should we constrict our focus on triangles? Is it true that if there are enough people at a party, there will always be at least k people who know each other or k of them who do not know each other? The following theorem lets us know that the answer to our prior question is in the affirmative.

Theorem 4.1 (Ramsey's Theorem)**.** *Let m and n be two positive integers, both of which are at least two. Then there exists a (minimal) positive integer $r(m, n)$ so that if we color the edges of a complete graph with $r(m, n)$ vertices red and blue, then this graph will either have a K_m subgraph with only blue edges or a K_n subgraph with only red edges [2].*

We know that any non-empty set of positive integers has a smallest element; therefore, if we can show that there exists at least one positive integer with our desired property, then we will have shown that a smallest such integer exists.

Proof. Let's have $m, n \geq 2$. We will prove this theorem using mathematical induction on m and n. First, we will prove the initial conditions that $r(m, 2)$ and $r(2, n)$ exists for all m and n. We then will take our induction step, proving that if $r(m, n-1)$ and $r(m-1, n)$ exist, then $r(m, n)$ also exists.

In order to see that our base case holds, we should note that $r(m, 2) = m$ and similarly, $r(2, n) = n$. Clearly, either all edges of K_m are colored blue and we have a K_m subgraph with all blue edges, or at least one of the edges is colored red, in which case it has a K_2 subgraph with all edges red. A similar argument can be used for $r(2, n)$.

We now move forward to our induction step, where we will prove that

$$r(m,n) \leq r(m,n-1) + r(m-1,n).$$

Let's consider an edge-coloring of a complete graph with $r(m,n-1)+r(m-1,n)$ vertices. Choose one of its vertices and call it V. We know that $d(V) = r(m,n-1)+r(m-1,n)-1$, meaning that V has either at least $r(m,n-1)$ red edges adjacent to it or $r(m-1,n)$ blue edges adjacent to it.

In our first case, let's take r to be the $r(m,n-1)-$element set of the other endpoints of these red edges. By definition of $r(m,n-1)$, our set r either contains a monochromatic blue K_m or a monochromatic red K_{n-1}, which can be completed to a monochromatic red K_n by adding vertex V. In either case, we have a desired monochromatic red or blue clique.

In our second case, let's take b to be the $r(m-1,n)-$element set of the other endpoints of these blue edges. Then again, b either contains a monochromatic red K_n or a monochromatic blue K_{m-1}, which can be completed to a monochromatic blue K_m by adding vertex V. In both cases, we again have a desired monochromatic red or blue clique.

We now see that $r(m,n-1)$ and $r(m-1,n)$ both exist, meaning that $r(m,n) \leq r(m,n-1) + r(m-1,n)$ is true. Thus, our induction step is also proved true. Therefore, we can say that there exists a minimal positive integer $r(m,n)$ that guarantees us either a blue K_m or a red K_n.

□

Now that we proved that the numbers taking the form $r(m,n)$ exist, we can formally define what they are.

Definition 4.2. Given two positive integers m and n we define the **Ramsey number** $r(m, n)$ to be the smallest p such that if we arbitrarily color the edges of the complete graph K_p blue or red, then we must have a blue K_m or a red K_n as a subgraph [3].

For example, the party problem gave us the value of the Ramsey number $r(3, 3) = 6$. We found that six vertices was the least number of vertices that guarantees us a blue or red K_3 when arbitrarily coloring our K_6.

We should note a few important properties of Ramsey numbers before progressing any further.

Proposition 4.3. *For any* $m, n \geq 2$, $r(m, n) = r(n, m)$.

Because we are arbitrarily coloring our complete graph, it will not matter whether we are trying to find a blue K_m or red K_n, or a red K_m or blue K_n. We are working with the same m and n in both cases, and because our coloring is arbitrary, we easily could change our colors around and get the same answer.

Proposition 4.4. *For any* $n \geq 1$, $r(1, n) = 1$.

We know that $r(1, n)$ is the smallest number of vertices such that when the edges of our complete graph are colored any two colors, we are guaranteed a complete subgraph of size one or a complete subgraph of size n. A complete subgraph of size one is just a single vertex. Thus, any graph with at least one vertex satisfies the conditions of having a a complete subgraph of size one. Therefore, we can say $r(1, n) = 1$.

The inequality we used in our induction step of the proof for Theorem 4.1 is an important result. Therefore, we state it in the following corollary.

Corollary 4.5. *For all positive integers $m, n \geq 2$, the inequality*

$$r(m, n) \leq r(m, n-1) + r(m-1, n)$$

holds.

Example 4.6. $r(3, 4) = 9$.

For this, we need to show that when arbitrarily coloring a complete graph blue and red, the smallest complete graph guaranteeing us a blue K_3 or a red K_4 is K_9. By Corollary 4.5, we know that $r(3, 4) \leq r(2, 4) + r(3, 3) = 4 + 6 = 10$. Next, observe this start of a complete coloring of K_8:

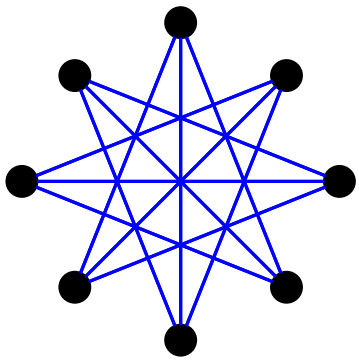


Figure 12: Partial coloring of K_8, showing blue edges.

By the symmetry of the figure, we see that no blue K_3 is present. Now color the remaining edges red to see that a red K_4 is not possible, for when you choose any four vertices, there is at least one blue edge among them. Thus, we now know that $r(3, 4) \geq 9$. Combining this with our upper bound, we can say $9 \leq r(3, 4) \leq 10$.

Let's first check to see if we are guaranteed a blue K_3 or a red K_4 when arbitrarily coloring a K_9 graph. To do this, let's focus our attention on vertex v_1 of Figure 13. Because its degree is eight, we know that there must be at least four edges of one color connecting v_1 to them; thus, without loss of generality, let's assume there are four blue edges connecting v_1 to four other vertices.

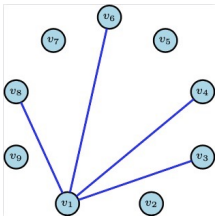


Figure 13: Blue edges from v_1

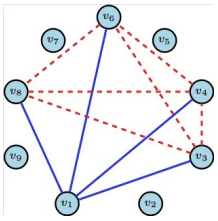


Figure 14: Red edges between end vertices

If any of the vertices at the ends of our blue edges are connected with another blue edge, then we will have a desired blue K_3. So, consider when all the vertices at the ends of our blue edges are connected with red edges. Well, this results in Figure 14 and we have a desired red K_4. Thus, it can be said that we are guaranteed a blue K_3 or red K_4 when arbitrarily coloring a K_9 graph. Using this and our bounds, we know that $r(3,4) = 9$.

Using this, let's try to find an estimate for $r(3,5)$. By our corollary, we know that

$$r(3,5) \leq r(2,5) + r(3,4).$$

We also know that $r(2,5) = 5$ and $r(3,4) = 9$. Therefore, we have

$$r(3,5) \leq 5 + 9 = 14.$$

This gets us one step closer to finding the exact value of $r(3,5)$.

Example 4.7. $r(3,5) = 14$.

To do this, we will show that a complete graph K_{13} can be colored without any blue K_3 or red K_5 [5]. We first need the following definition:

Definition 4.8. Let $a, n \in \mathbb{Z}$ such that $a, n \geq 1$. Then a is a **cubic residue mod n** if there exists an x such that $x^3 \equiv a \bmod n$.

To make sure we understand the concept, let's consider some cubic residues mod 9. We will work with the set $\{1, 2, 4, 5, 7, 8\}$, for these are all numbers that are co-prime to 9. When cubing 1, 4, and 7, we obtain numbers that are equivalent to 1 mod 9. When cubing 2, 5, and 8, we obtain numbers that are equivalent to 8 mod 9. Thus, we can say that 1 and 8 are cubic residues mod 9. The numbers 5 and 7 would be examples of cubic non-residues.

Let's consider the complete graph K_{13} with vertices labeled 0,1,...,12, all of which are co-prime with 13. When cubing 1, 3, and 9, we obtain numbers that are equivalent to 1 mod 13. When cubing 7, 8, and 11, we obtain numbers equivalent to 5 mod 13. When cubing 2, 5, and 6, we obtain numbers equivalent to 8 mod 13. And finally, when cubing 4, 10, and 12, we get numbers equivalent to 12 mod 13. Thus, the cubic residues of these labeled vertices are 1, 5, 8, and 12. These values will give us some guidelines for the coloring of our graph. If the difference between two vertices is a cubic residue, then we color the corresponding edge blue. If the difference is not a cubic residue, then we color the corresponding edge red. Note that because $12 \equiv -1 \bmod 13$, the order of our subtraction does not matter. By these rules, we obtain the following two subgraphs of K_{13}:

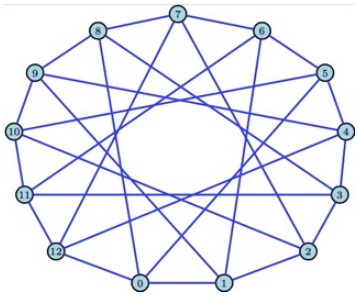


Figure 15: Blue edges of K_{13} coloring

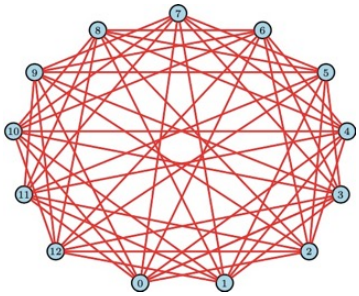


Figure 16: Red edges of K_{13} coloring

Observing Figure 15, we see that when choosing any three vertices, there will always be a red edge among them. Therefore, we have no blue K_3. Observing Figure 16, we see that when choosing any five vertices, there will always be a blue edge among them. By this, we also have no red K_5. Thus, we know that $r(3,5) > 13$. Combining this with our previous upper bound for $r(3,5)$, it can be said that $r(3,5) = 14$, which is what we intended to show.

Before moving forward to proving some bounds, we look at the computation of one more Ramsey number.

Example 4.9. $r(4,4) = 18$.

Showing this relies on the following definition:

Definition 4.10. Let $a, n \geq 1$. Then a is a **quadratic residue mod n** if there exists an x such that $x^2 \equiv a \bmod n$.

Similar to our last example, let's consider some quadratic residues mod 9 to make sure we understand the concept. Again, we will work with the set {1,2,4,5,7,8}. When squaring 1 and 8, we obtain numbers that are equivalent

to 1 mod 9. When squaring 2 and 7, we obtain numbers that are equivalent to 4 mod 9. And finally, when squaring 4 and 5, we obtain numbers that are equivalent to 7 mod 9. Thus, we can say that 1, 4, and 7 are quadratic residues mod 9. The numbers 2 and 8 would be examples of quadratic non-residues.

By Theorem 4.6, we know that $r(4,4) \leq r(3,4) + r(4,3) = 18$. Thus, if we can show that a coloring of a complete graph K_{17} does not guarantee us a blue or red K_4, we will be able to find our answer [5].

To do this, let's consider vertices numbered from 0 to 16, inclusive. If two of these vertices have a difference numerically that is a quadratic residue of 17, then we color the edge connecting the two vertices red. Edges that connect two vertices that do not have a difference of a quadratic residue are to be colored blue. We should note that because $-1 \equiv 16 \pmod{17}$, the order in which our subtraction is performed does not make a difference.

Now, suppose that four vertices are connected by edges of one color, and without loss of generality, we can say that one of these four is the vertex labeled 0. We call the other three vertices a, b, c. Then, the six numbers $a,\ b,\ c,\ a-b,\ a-c,\ b-c$ are all either residues or non-residues. It's now important to note that because a is not the zero vertex in our field, multiplying by a^{-1} is allowed.

So, let's set $B = ba^{-1}$ and $C = ca^{-1}$ and consider the new set of six numbers $1,\ B,\ C,\ 1-B,\ 1-C,\ B-C$. As 1 is one of these six numbers, we know that all six must be residues of 17. However, we can compute the quadratic residues mod 17 to be $1, 2, 4, 8, 9, 13, 15$, and 16, and no selection of values for B and C can make every element in the set a quadratic residue.

This arrives us at a contradiction, meaning that $r(4,4) > 17$.

Using Theorem 4.6, we can establish $r(4,4) \leq 18$; thus, given we just proved that $r(4,4) > 17$, it can be said that $r(4,4) = 18$.

Up to this point, we have found our fair share of Ramsey numbers. To see our progress, let's observe the following table:

m, n	1	2	3	4	5
1	1	1	1	1	1
2	1	2	3	4	5
3	1	3	6	9	14
4	1	4	9	18	$r \leq 32$
5	1	5	14	$r \leq 32$	$r \leq 64$

Figure 17: Ramsey numbers $r(m,n)$ so far

While this is good progress for the theorems we have stated so far, there is always room to grow. So, we now move forward to proving more bounds, allowing us to expand our table even further.

5 A Couple of Bounds

So far, we have proved only one upper bound for Ramsey numbers, Corollary 4.5. In order to find more accurate estimates of Ramsey numbers, we need to find other bounds to work with, both upper and lower. Some bounds work with any Ramsey number $r(m, n)$, while others work specifically with numbers taking the form $r(m, m)$. The next bound we will look at requires the use of Corollary 4.5 and the following identity.

Remark 5.1 (Pascal's Identity)**.** For all positive integers n, k with $k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Theorem 5.2. *For all $m, n \geq 1$, $r(m, n) \leq \binom{m+n-2}{m-1}$.*

Proof. This theorem can be proved using induction on the quantity $m + n$. In looking at our base cases, we can observe the following:

$$\begin{aligned} r(1, n) &= 1 = \tbinom{1+n-2}{1-1}, \\ r(m, 1) &= 1 = \tbinom{m+1-2}{m-1}, \\ r(2, n) &= n = \tbinom{2+n-2}{2-1}, \\ r(m, 2) &= m = \tbinom{m+2-2}{m-1}. \end{aligned}$$

Thus, because our base cases hold true, we can assume our result holds for smaller values of m and n. We have proved $r(m, n) \leq r(m-1, n) + r(m, n-1)$ for $m, n > 2$.

Therefore, using this and Pascal's Identity,

$$\begin{aligned} r(m,n) &\leq r(m-1,n) + r(m,n-1) \\ &\leq \binom{(m-1)+n-2}{(m-1)-1} + \binom{m+(n-1)-2}{m-1} \\ &= \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} \\ &= \binom{m+n-2}{m-1}. \end{aligned}$$

Thus, we have $r(m,n) \leq \binom{m+n-2}{m-1}$.

□

Using Theorem 5.1, we are able to arrive at the following corollary:

Corollary 5.3. *For all integers $m \geq 2$, the inequality $r(m,m) \leq 4^{m-1}$ holds.*

A proof of this is straightforward. We know that a set of a elements has 2^a different subsets, for when thinking about the elements of this set, each one is either in the given subset or not. Thus, because $\binom{a}{b}$ represents how many different ways we can choose b elements from a set of a elements, it can be said that $\binom{a}{b} \leq 2^a$. By Theorem 5.1, we know that $r(m,m) \leq \binom{2m-2}{m-1}$. Hence, we have

$$r(m,m) \leq \tbinom{2m-2}{m-1} \leq 2^{2m-2} = 4^{m-1}.$$

This shows that our desired inequality holds.

We now turn our attention to finding some lower bounds for Ramsey numbers using probabilistic methods. To avoid confusion, let us note that

P denotes probability and bar denotes negation. The following result, found in [11], is needed to proceed:

Lemma 5.4. *Let $A_i, 1 \leq i \leq m$, be events in a probability space with $P(A_i) \leq p$. If $mp < 1$, then*

$$P(\bar{A}_1 ... \bar{A}_m) > 0.$$

The quantity $\bar{A}_1 ... \bar{A}_m$ is the product of all the compliments for each A_i with $1 \leq i \leq m$. When our conditions are satisfied, and $P(\bar{A}_1 ... \bar{A}_m)$ is greater than zero, it can be said that the probability of each individual compliment occurring, $\bar{A}_i$, is also greater than zero. Therefore, it can be said that no A_i is guaranteed to occur. Using this, we look at a theorem also found in [11].

Theorem 5.5. *If*

$$\binom{n}{m} 2^{1-\binom{m}{2}} < 1,$$

then $r(m, m) \geq n$.

Proof. Let's fix m, n satisfying our given inequality, and have G be a random 2-coloring of K_n. By random, we mean that each edge is independently colored with equal probability of either color (red or blue) occurring. If we call S a set of m vertices, let A_S be the event in which all edges on S are the same color. Given a particular such set S, we see that when coloring all $\binom{m}{2}$ edges blue or all red, we obtain

$$P(A_S) = \frac{1}{2^{\binom{m}{2}}} + \frac{1}{2^{\binom{m}{2}}} = \frac{2}{2^{\binom{m}{2}}} = 2^{1-\binom{m}{2}}.$$

We should also note that there are $\binom{n}{m}$ different ways to choose S from our K_n. Now, doing a union bound over all possible selections of S, and using our original assumption, we arrive at the following inequality.

$$P(\text{at least one } A_S) \leq \sum_S P(A_S) = \binom{n}{m} 2^{1-\binom{m}{2}} < 1.$$

Given this, we see that

$$P(\text{no } A_S \text{ happens}) = 1 - P(\text{at least one } A_S) > 0.$$

Because the probability of no A_S is greater than zero, it cannot be said that we are guaranteed a blue or red K_m with n vertices. Therefore, we have our desired inequality $r(m, m) \geq n$.

□

By generalizing this theorem, we obtain a lower bound, found in [11], for Ramsey numbers that take the form $r(m, n)$.

Theorem 5.6. *If there exists a* p, $0 \leq p \leq 1$, *such that*

$$\binom{n}{m} p^{\binom{m}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

then $r(m, t) > n$.

Proof. Let's fix p, t, m, n satisfying the given inequality. We claim that $r(m, t) > n$. Using blue and red, let $\mathbf{G}$ be a two coloring of K_n. Each edge has probability p of being blue and $(1 - p)$ of being red, with all of these probabilities being mutually independent.

If we call M a set that has m vertices, let's call A_M the event that all edges of M are colored blue. Similarly, if we call T a set that has t vertices,

let's call B_T the event that all edges of T are colored red. Note that there are $\binom{n}{m}$ ways to choose M from our K_n and $\binom{n}{t}$ ways to choose T form our K_n.

Taking any possible set M, the edges between all m vertices have a possibility of two colors; thus, the probability all edges are blue is $p^{\binom{m}{2}}$. A similar argument can be used to show that when taking any possible set T, the probability all edges are red is $(1-p)^{\binom{t}{2}}$.

Given this information, we know that the total probability of getting a desired monochromatic subgraph is

$$\binom{n}{m}p^{\binom{m}{2}} \ + \ \binom{n}{t}(1-p)^{\binom{t}{2}}.$$

From our assumptions, we know this sum is less than one. Therefore, we are not guaranteed a monochromatic subgraph with only n vertices; thus, we have the desired inequality $r(m,t) > n$.

□

Let's look at a couple of examples to see how our previous two theorems can be of help.

Example 5.7. Using Theorem 5.4, we find a lower bound for $r(5,5)$.

Given that we are trying to find a lower bound for $r(5,5)$, our m-value is five. Thus, we are looking at the following:

$$\binom{n}{5}2^{1-\binom{5}{2}} < 1.$$

Doing some basic arithmetic, we transform this into

$$\binom{n}{5} < 2^9 = 512.$$

We find the smallest n- value making this inequality true to be 11; thus, it can be said that $r(5,5) \geq 11$.

Example 5.8. Using Theorem 5.5, we find a lower bound for $r(4,5)$.

Our $m-$value is four and our $t-$value is five, meaning we need to find a $p-$ and $n-$value that satisfy the inequality

$$\binom{n}{4}p^{\binom{4}{2}} + \binom{n}{5}(1-p)^{\binom{5}{2}} < 1.$$

Observe that when we take $n = 8$ and $p = \frac{2}{5}$, and do some simple computations, we have

$$70(\tfrac{2}{5})^6 + 56(\tfrac{3}{5})^{10} < 1.$$

Thus, by Theorem 5.5, it can be said that $r(4,5) > 8$.

These prior two examples show us that while Theorems 5.4 and 5.5 give us a lower bound, the bound is very weak.

We can prove another lower bound for Ramsey numbers using a non-probabilistic method, and the process is shown in the proof of the following theorem [9].

Theorem 5.9. *For* $m, n > 0$, $r(m,n) > (m-1)(n-1)$.

Proof. Let's have $m, n > 0$. We claim that $r(m,n) > (m-1)(n-1)$. To prove this claim, let's have $s = (m-1)(n-1)$. What we are ultimately trying to show now is that a 2-coloring of a K_s graph does not guarantee a blue K_m or a red K_n subgraph.

Arrange the vertices of our K_s in $m-1$ rows and $n-1$ columns. If two vertices are in the same row, we will connect them with a red edge. All other

edges can be colored with a blue edge. If this coloring does not include a blue K_m or red K_n, then our theorem will be proven true.

In regard to having a blue K_m, we obviously will need m vertices. But, we only have $m-1$ rows, meaning two vertices would have to be in the same row. Because vertices in the same row are colored with a red edge, we can confirm that our coloring does not include a blue K_m.

Similarly, in regard to having a red K_n, we need n vertices. But, we only have $n-1$ columns, meaning two of our vertices need to be in the same column. Because vertices in the same column are colored with a blue edge, we confirm that our coloring also does not include a red K_n.

We have shown our coloring of K_s does not include a blue K_m or a red K_n, meaning $r(m,n) > s$. Since we set $s = (m-1)(n-1)$, we can now say that $r(m,n) > (m-1)(n-1)$. □

Example 5.10. Using the method from Theorem 5.6, $r(3,4) > 6$.

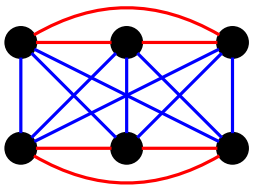


Figure 18: Theorem 5.6 example using $r(3,4)$.

In the figure, we see we are working with a K_6, for $(m-1)(n-1) = 2(3) = 6$. With the vertices arranged in two rows and three columns, we connect vertices in the same row with red edges. All other edges are colored blue. In observing the colored figure, it is clear that there does not exist a blue K_3 or a red K_4; thus, it can be said that $r(3,4) > 6$.

We have already found the Ramsey number $r(3,4)$ to be equal to nine, so this example does not help us further our research. However, it can be used to improve prior bounds found using different methods. For example, using Theorem 5.4, we previously showed that $r(5,5) \geq 11$. Now, using Theorem 5.8, it can be said that $r(5,5) > 16$.

It's important to note that these bounds are still relatively weak, which shows how complicated of a topic Ramsey numbers are. With the bounds we currently have proved, let's update our table:

m, n	1	2	3	4	5
1	1	1	1	1	1
2	1	2	3	4	5
3	1	3	6	9	14
4	1	4	9	18	$12 < r \leq 32$
5	1	5	14	$12 < r \leq 32$	$16 < r \leq 64$

Figure 19: Updated table of Ramsey numbers known

This table is a great example of how weak our established lower bounds are. We have shown that $r(3,5) = 14$, yet our lower bound for $r(4,5)$ is found to be 12 using Theorem 5.8. Similar arguments can be used for $r(5,5)$.

To find strong bounds for these numbers and dive deeper into the Ramsey realm, we need techniques significantly more advanced than the ones we have studied so far, such as expansions of the probabilistic method. Instead of studying these, we will turn our focus to Ramsey numbers of a different form: ones that include more than two colors.

6 Multicolor Ramsey Numbers

Not all Ramsey numbers focus only on two colors. Some focus on three or more, and we call these **multicolor** Ramsey numbers. Similar to Ramsey numbers that deal with only two colors, we must first prove that these multicolor Ramsey numbers exist [2].

Theorem 6.1. *Let $n_1, n_2, ..., n_k$ be positive integers, with k fixed. Then there exists a minimal positive integer $N = r(n_1, n_2, ..., n_k)$ so that if $n \geq N$, and we color all edges of $G = K_n$ with colors $1, 2, ..., k$, then there will always be at least one index $i \in 1, 2, ..., k$ so that G has a K_{n_i} subgraph whose edges are all of color i.*

Proof. Our proof can be done by induction on the sum $n_1 + n_2 + ... + n_k$. The base case of $n_1 = n_2 = ... = n_k = 1$ is considered trivial. Whenever dealing with a Ramsey number taking on the form $r(1, 1, ..., 1)$, a single vertex guarantees us a K_1 subgraph of any color. So, we can assume that we know the statement for all positive integers $n_1, n_2, ..., n_k$ whose sum is *less than* m, and prove it for the case when their sum is equal to m.

Let's have $n_1, n_2, ..., n_k$ be positive integers. By our induction hypothesis, we assume that the positive integer $r(n_1 - 1, n_2, ..., n_k)$ exists. Let's set $N = k(r(n_1 - 1, n_2, ..., n_k) - 1) + 2$, and assume that our graph $G = K_n$ has a vertex V so that the color that occurs most frequently among the edges adjacent to V is color 1. This means that at least $r(n_1 - 1, n_2, ..., n_k)$ edges adjacent to V are of color 1. Now call S the set of endpoints of these edges, other than V, and let K_S be the complete graph with vertex set S.

By the definition of $r(n_1 - 1, n_2, ..., n_k)$, we know there either exists a

vertex $i \in 2, 3, ..., k$ so that K_S has a K_{n_i} subgraph with all edges colored i, or K_S has a K_{n_1-1} subgraph with all edges colored 1. When adding the vertex V to both of these cases, we see that we have our desired subgraphs.

To understand the general case, we proceed in a similar way to before, except for our choice of N. Set $N = r(n_1 - 1, n_2, ..., n_k) + r(n_1, n_2 - 1, n_3, ..., n_k) + ... + r(n_1, n_2, ..., n_k - 1) - k + 2$. It follows by the pigeonhole principle that there exists an $i \in [k]$ so that there are at least $r(n_1, ..., n_i - 1, ..., n_k)$ edges adjacent to our vertex V that are colored i.

Call S the set of endpoints of these edges, again not including V, and let K_S be the complete graph with vertex set S. By our definition of $r(n_1, ..., n_i - 1, ..., n_k)$, we know there either exists some vertex $j \in 1, 2, ..., i-1, i+1, ..., k$ so that K_S has a K_{n_j} subgraph with all edges colored j, or K_S has a K_{n_i-1} with all edges colored colored i. When adding V to both of these cases, we have our desired subgraphs, meaning our proof is complete.

□

Now that we have proven the existence of these numbers, we can formally define them.

Definition 6.2. Given positive integers $n_1, n_2, ...,$ and n_k, we define the multicolor Ramsey number $r(n_1, n_2, ..., n_k)$ to be the smallest p such that if we arbitrarily color the edges of the complete graph K_p color 1, color 2,..., or color k, then we must get a 1-colored K_{n_1}, a 2-colored K_{n_2},..., or a k-colored K_{n_k} as a subgraph.

As of now, there are only two known multicolor Ramsey numbers due to the nature and complexity of them: $r(3, 3, 3)$ and $r(3, 3, 4)$. We will first

look at $r(3,3,3)$ and prove its value, then move forward and study $r(3,3,4)$.

Example 6.3. $r(3,3,3) = 17$.

We show this by giving the proof in two parts. First, we observe that $r(3,3,3) > 16$. In [5], Greenwood and Gleason use properties of cubic residues to show that no monochromatic triangle is guaranteed in a complete graph with 16 vertices. Using this, we now must show that a complete graph with 17 vertices is sufficient to guarantee us a red, blue, or green triangle.

In any 3-coloring of a graph with 17 vertices, let's focus on one specific vertex and call it v. Of all the edges terminating at v, we know at least six of them must be the same color. Without loss of generality, let there be six blue edges terminating at v, resulting in the following figure.

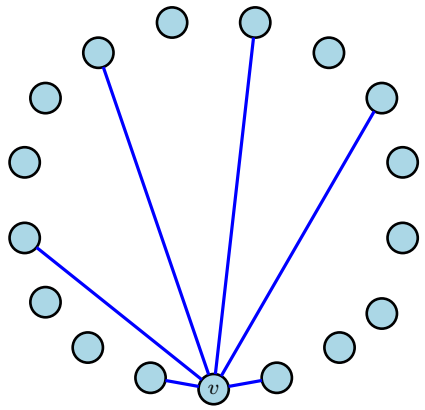


Figure 20: Complete graph K_{17} with vertex v.

If the vertices at the other ends of these six blue edges have at least one blue edge between them, then we have a blue K_3 as desired. So, consider when these vertices have no blue edges between them. Well, that means all edges are colored either red or green, resulting in the $r(3,3)$ case. We

have previously shown that we are guaranteed a monochromatic triangle when coloring K_6 with two colors. Thus, we can say in either case, we are guaranteed a monochromatic blue, red, or green triangle. Therefore, we have shown that $r(3,3,3) = 17$.

The multicolor Ramsey number $r(3,3,4)$ is significantly more complicated. In 1966, James Kalbfleisch [6] proved that $r(3,3,4) \geq 30$. It wasn't until 32 years later that the upper bound of 31 was found by Konrad Piwakowski and Stanislaw Radziszowksi [8]. Thus, with these findings we have

$$30 \leq r(3,3,4) \leq 31.$$

In 2016, Codish, Frank, Itzhakov, and Miller were able to use advanced techniques to find the exact value of $r(3,3,4)$ to be 30. These techniques, formally known as *abstraction* and *symmetry breaking*, are far beyond the level of this paper. To learn more about these techniques, and to see the computations made to find $r(3,3,4) = 30$, see [4].

7 Ramsey Theory in Geometry

We know that the number of points in the two-dimensional plane is infinite. Given that we have been focused only on finite graphs, the following examples will certainly be different than those observed so far. In what follows, we will prove some statements of geometric nature, giving us some insight on the realm of *combinatorial geometry*.

Example 7.1. Let us assume *all points of the plane* are colored either red or blue. There exists a unit segment with monochromatic endpoints [2].

Proof. Assuming that all points of the plane are colored red or blue, let's take a regular triangle T with side length 1. By the pigeon-hole principle, we know that our triangle T must have at least two vertices of the same color. To visualize this, consider the following three cases:

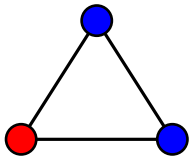


Figure 21: Case One

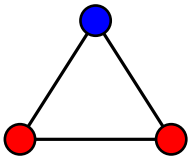


Figure 22: Case Two

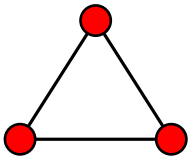


Figure 23: Case Three

In Case One, we have one unit segment with blue endpoints. In Case Two, we have one unit segment with red endpoints. And finally, in Case Three, we have three unit segments with red endpoints. Each case presents

us with at least one unit segment with monochromatic endpoints, which completes our proof. □

We can strengthen this statement in a similar way to how we strengthened our arguments for Ramsey numbers (from two colors to multicolor).

Example 7.2. Let us assume that *all points of the plane* are colored red, blue, or green. There exists a unit segment with monochromatic endpoints [2].

Proof. Consider the circle centered at A with a radius of length $\sqrt{3}$, and without loss of generality, say that A is colored green.

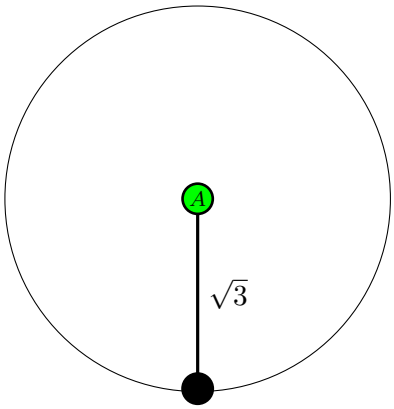


Figure 24: Circle w/ radius $\sqrt{3}$ centered at A.

When the points on this circle are all colored green, it is easy to see how we could find a chord of unit length with monochromatic (green) endpoints. So, let's considered when the points on the circle are not all colored green. Consider the point B on our circle, and without loss of generality, let's say that B is colored red.

Since AB is of length $\sqrt{3}$, we can use some geometric methods and properties to construct two regular triangles with sides of unit length: ΔACD and ΔBCD.

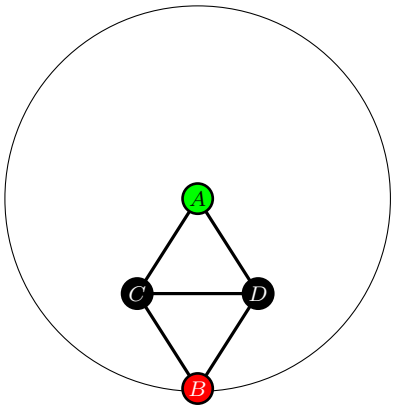


Figure 25: Constructed regular triangles ACD and BCD.

If either C or D are colored red or green, we have our desired segment of unit length with monochromatic endpoints. So consider when both C and D are colored blue. Then we also have a segment of unit length with monochromatic (blue) endpoints. Therefore, we see that whether or not our circle is colored monochromatically does not matter, and we are always guaranteed a segment of unit length with monochromatic endpoints.

□

These few examples were a short dive into combinatorial geometry, mainly serving the purpose to show how Ramsey theory methods can be used in this complicated subject. To dive deeper, the reader is encouraged to study [7].

8 Examples using Ramsey Methods

To highlight some of the methods we have studied, and proved, throughout our research, we look at some examples. Our first will utilize the complete graph K_{18} to prove a statement related to a popular sport.

Example 8.1. Eighteen teams participate at a round-robin soccer tournament. After eight rounds are played, we can still find three teams no two of which have played each other yet [2].

As previously mentioned, let's observe a complete graph K_{18} whose vertices correspond to eighteen soccer teams. After eight rounds are played, we color the edge between two teams red if they have played each other and blue if they have not. Our goal is to show that we are guaranteed a blue triangle in our graph.

Take any team (vertex) A and consider the nine teams A has not yet played.

If any two of these nine teams have also not played each other, say B and E, then ABE is a blue triangle and we have our desired monochromatic subgraph.

Consider the case when there are no such two teams. This would mean that any two of the nine teams that have not played A have played each other. In other words, these teams have completed a round-robin tournament among themselves. However, this is impossible for nine teams in only eight rounds. In one round, they could only play four games among themselves, meaning in eight rounds, they could play at most 32 games. This is less than the total number of $\binom{9}{2} = 36$ games needed for a round-robin

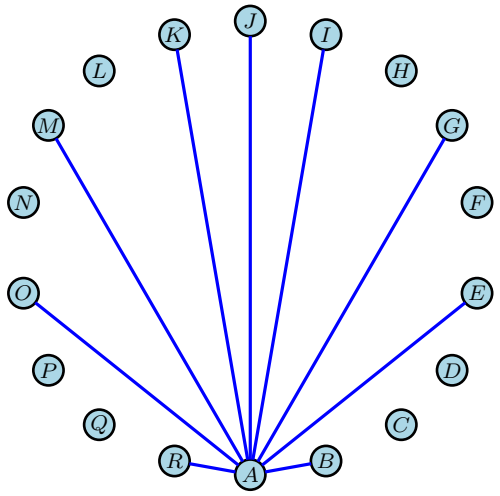


Figure 26: Teams A has NOT played.

tournament with nine teams.

Thus, it is guaranteed that at least two of these nine teams have also not played each other, and we are guaranteed a monochromatic blue triangle.

The next example we will look at closely relates to our previous geometry section.

Example 8.2. Let ABC be a regular triangle, and let E be the set containing all points of the closed segments AB, AC, and BC. We color each point of E red or blue. No matter what coloring we choose, there will always be a right-angled triangle with monochromatic vertices [2].

To solve this exercise, let's have C_1 and C_2 be the points that divide the segment AB into three equal parts. A_1, A_2, B_1, and B_2 can be defined in a similar way. By this, we have the following triangle.

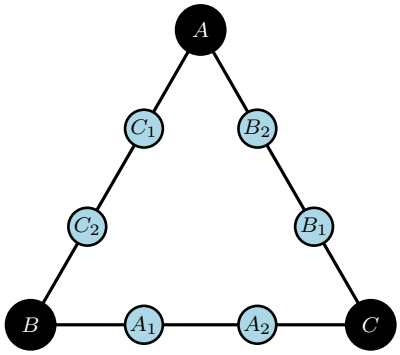


Figure 27: Triangle setup for Example 8.3.

At least two of A_1, B_1, and C_1 are of the same color; thus, without loss of generality, let's say that A_1 and B_1 are both red. Now, assume there is no right-angled triangle with monochromatic vertices. In connecting A_1 and B_2 with an edge, we see that triangle A_1B_2C is also equilateral. By properties of equilateral triangles, an edge drawn between A_1 and B_1 would be the perpendicular bisector of side B_2C. This creates two right-angled triangles: $A_1B_1B_2$ and A_1B_1C. Thus, from our prior assumption, C and B_2 must both be blue and we arrive at the figure below.

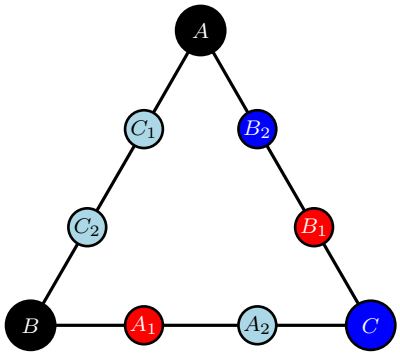


Figure 28: Colored vertices of Example 8.3 Triangle

Since we assumed there are no right-angled triangles with monochromatic vertices, we cannot find a color for C_2. If C_2 is red, we get the following figure.

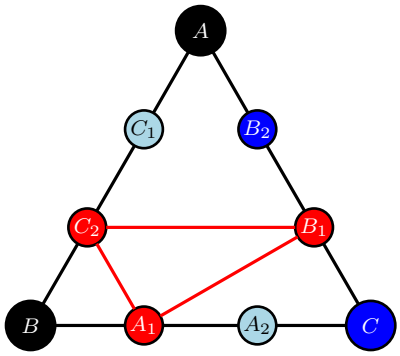


Figure 29: Right-angled red triangle $A_1B_1C_2$.

We have already determined the degree of angle CA_1B_1 to be 30, and since triangle A_1BC_2 is equilateral, we know angle BA_1C_2 is 60. Therefore, angle $C_2A_1B_1$ must be a right angle, meaning triangle $A_1B_1C_2$ is right-angled with three monochromatic (red) vertices. Similar arguments can be made to show that angle C_2B_2C is also a right angle. This implies that vertex C_2 cannot be colored blue because triangle CB_2C_2 would then be right-angled with monochromatic (blue) vertices. Since we are unable to find a color for C_2, our assumption that there are no right-angled triangles with monochromatic vertices is false.

Our final example takes us back to our original Ramsey party. This time, instead of showing we are guaranteed at least one group of three that either know or don't know each other, we will show that at least two of these groups exist in a party of six.

Example 8.3. If we color the edges of K_6 red or blue, there will be at least *two* triangles with monochromatic edges.

Consider the complete graph on six vertices K_6. From our original party problem, we know that we are guaranteed at least one monochromatic triangle within these six vertices. Without loss of generality, say that this triangle is blue and call it Δ.

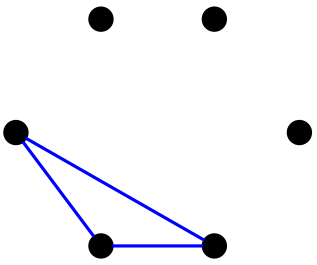


Figure 30: Monochromatic triangle Δ with blue edges.

Now, consider the other three vertices not included in Δ. If the edges between them are all blue, then we have found our second desired monochromatic triangle. So, consider when one of these edges is red and call it AB.

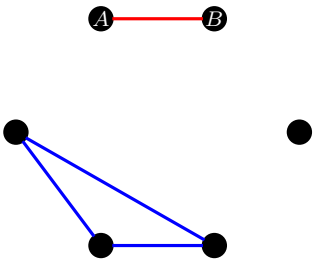


Figure 31: Red edge AB that is NOT in Δ.

If there are two blue edges going from A to Δ, then we have our second blue triangle. So, assume there are two red edges going from A to Δ.

Similarly, say there are two red edges going from B to Δ. This means there is a C in Δ such that AC and BC are both red. Again, without loss of generality, consider the following figure.

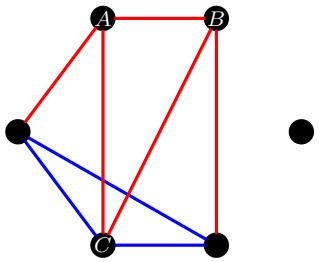


Figure 32: Connecting A and B to Δ w/ red edges.

Thus, we see that triangle ABC is also monochromatic, meaning we have shown that at least two monochromatic triangles are guaranteed when coloring the edges of K_6 red or blue.

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