GENERATORS FOR DECOMPOSITIONS OF TENSOR PRODUCTS OF MODULES ASSOCIATED WITH STANDARD JORDAN PARTITIONS

MICHAEL J. J. BARRY

Abstract. If K is a field of finite characteristic p, G is a cyclic group of order $q = p^\alpha$, U and W are indecomposable KG-modules with $\dim U = m$ and $\dim W = n$, and $\lambda(m,n,p)$ is a standard Jordan partition of mn, we describe how to find a generator for each of the indecomposable components of the KG-module $U \otimes W$.

1. Introduction

Let p be a prime number, K a field of characteristic p, and G a cyclic group of order $q = p^\alpha$, where α is a positive integer. It is well-known that there are exactly q isomorphism classes of indecomposable KG-modules and that such modules are cyclic and uniserial [1, p. 24–25]. Let $\{V_1, \ldots, V_q\}$ be a set of representatives of these isomorphism classes with $\dim V_i = i$. Many authors have investigated the decomposition of the KG-module $V_m \otimes V_n$, where $m \le n$, into a direct sum of indecomposable KG-modules—for example, in order of publication, see [9], [16], [12], [13], [15], [10], [14], [11], and [3]. From the works of these authors, it is well-known that $V_m \otimes V_n$ decomposes into a direct sum $V_{\lambda_1} \oplus V_{\lambda_2} \cdots \oplus V_{\lambda_m}$ of m indecomposable KG-modules where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m > 0$, but that the dimensions λ_i of the components depend on the characteristic p. Now $\lambda(m,n,p) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is called a **Jordan partition** of mn, and $\lambda(m,n,p)$ is said to be **standard** exactly when $\lambda_i = m + n + 1 - 2i$ for every integer $i \in [1,m]$. So when $\lambda(m,n,p)$ is standard,

$$V_m \otimes V_n \cong \bigoplus_{i=1}^{m} V_{n+m+1-2i}.$$

Necessary and sufficient conditions on m, n, and p for $\lambda(m,n,p)$ to be standard were given in [5].

Fix a generator g of G. There is a basis $u_1, u_2, \ldots, u_m$ of V_m on which the action of g is given by $gu_1 = u_1$ and $gu_i = u_{i-1} + u_i$ when $i > 1$. Note that $(g-1)^i u_m = u_{m-i}$, and so u_m generates V_m as a KG-module. Similarly there is a basis $w_1, w_2, \ldots, w_n$ of V_n, with V_n generated as a KG-module by w_n, on which the action of g is given by $gw_1 = w_1$ and $gw_i = w_{i-1} + w_i$ when $i > 1$. Clearly $\{v_{i,j} = u_i \otimes w_j \mid 1 \le i \le m, 1 \le j \le n\}$ is a basis of $V_m \otimes V_n$ over K. Shortly

1991 *Mathematics Subject Classification.* Primary 20C20.
Key words and phrases. Tensor product, indecomposable module, cyclic group, module generator.

we will see that $\mathcal{B} = \{f_{i,j} = u_i \otimes g^{n-i} w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is also a basis of $V_m \otimes V_n$ that turns out to be easier to work with. We will specify, in terms of $\mathcal{B}$, m elements y_1, y_2, ..., y_m in $V_m \otimes V_n$ such that, when $\lambda(m, n, p)$ is standard, $KGy_i \cong V_{n+m+1-2i}$ and $V_m \otimes V_n$ is an internal direct sum of the indecomposable modules KGy_i. $(m \geq 2)$

Barry did this for the special standard partition $\lambda(m, n, p)$ with $m + n \leq p + 1$ in [4], and Glasby, Praeger, and Xia did it for a subset of standard partitions that properly includes the case that Barry dealt with in [6, Theorem 2].

We now describe the organization of the paper. In Section 2, we show how to calculate in $V_m \otimes V_n$, we state our main result in Section 3, we work out an example in Section 4, and finally we prove our main result in Section 5.

2. Preliminaries

Lemma 1. *The set* $\mathcal{B} = \{f_{i,j} = u_i \otimes g^{n-i} w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ *is an* K*-basis for* $V_m \otimes V_n$*, and* $(g-1)f_{i,j} = f_{i-1,j} + f_{i,j-1}$*, where we understand that* $f_{k,\ell} = 0$ *if* $k < 1$ *or* $\ell < 1$.

Proof. Since $f_{i,j} = v_{i,j} + \sum_{k+\ell<i+j} \alpha_{k,\ell} v_{k,\ell}$, the linear independence of $\mathcal{B}$ follows from the linear independence of $\{v_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Also

$$\begin{aligned} gf_{i,j} &= gu_i \otimes g^{n-i+1} w_j \\ &= (u_i + u_{i-1}) \otimes g^{n-i}(w_j + w_{j-1}) \\ &= u_i \otimes g^{n-i} w_j + u_i \otimes g^{n-i} w_{j-1} + u_{i-1} \otimes g^{n-i}(w_j + w_{j-1}) \\ &= f_{i,j} + f_{i,j-1} + u_{i-1} \otimes g^{n-(i-1)} w_j \\ &= f_{i,j} + f_{i,j-1} + f_{i-1,j} \end{aligned}$$

Thus $(g-1)f_{i,j} = f_{i,j-1} + f_{i-1,j}$. □

Preference for working with the basis $\mathcal{B}$ over the basis $\{v_{i,j} = u_i \otimes w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ stretches back at least to Trampus in [17] and [18] .

For an integer $k \in [1, m+n-1]$, define the K-vector subspaces F_k and D_k of $V_m \otimes V_n$ by

$$F_k = \langle f_{i,j} \mid i + j \leq k + 1 \rangle, \text{ and } D_k = \langle f_{i,j} \mid i + j = k + 1 \rangle.$$

Also for convenience define $D_k = \{0\}$ for $k < 1$. Clearly $D_k \subset F_k$. Note that F_k, which equals $\langle v_{i,j} \mid i + k \leq k + 1 \rangle$, is a KG-submodule of $V_m \otimes V_n$.

Lemma 2. *For each integer* $i \in [1, m]$*, define* $x_i = \sum_{j=1}^{i} (-1)^{j-1} f_{j,i+1-j} \in D_i$*. Then* $\{x_1, x_2, \ldots, x_m\}$ *is linearly independent over* K *and* $(g-1)x_i = 0$ *for all* i.

Note that the definition of the elements x_i differs from that in [3].

Proof. First $\{x_1, x_2, \ldots, x_m\}$ is linearly independent over K because $x_i \in D_i$ and $F_m = D_1 \oplus \cdots \oplus D_m$ as a direct sum of vector spaces. Also

$$\begin{aligned}(g-1)x_i &= \sum_{j=1}^{i}(-1)^{j-1} f_{j,i-j} + \sum_{j=1}^{i}(-1)^{j-1} f_{j-1,i+1-j} \\ &= \sum_{j=1}^{i}(-1)^{j-1} f_{j,i-j} + \sum_{j=0}^{i-1}(-1)^{j} f_{j,i-j} \\ &= f_{i,0} + f_{0,i} \\ &= 0.\end{aligned}$$

□

Then, by Lemma 1, $(g-1)^r(D_k) \subseteq D_{k-r}$ and

$$(g-1)^r(f_{i,j}) = \sum_{k=0}^{r}\binom{r}{k} f_{i+k-r,j-k},$$

where we understand that $f_{i+k-r,j-k} = 0$ if $i+k-r < 0$ or $j-k < 0$.

Hence $(g-1)^{m+n-2k}(D_{m+n-k}) \subseteq D_k$ when $1 \le k \le m$ and

$$(g-1)^{m+n-2k}(f_{i,j}) = \sum_{\ell=0}^{m+n-2k}\binom{m+n-2k}{\ell} f_{i+\ell-m-n+2k,j-\ell}.$$

Assume that $m \le n$. Denote the ordered K-basis

$$(f_{m-k+1,n}, f_{m-k+2,n-1} \ldots, f_{m,n-k+1})$$

of D_{m+n-k} by $\mathcal{B}_{m+n-k}$ and the ordered K-basis $(f_{1,k}, f_{2,k-1}, \ldots, f_{k,1})$ of D_k by $\mathcal{B}_k$. In the case where $m = n$ and $k = m$, $\mathcal{B}_{m+n-k} = \mathcal{B}_k$.

Lemma 3. *Let $A_k(m,n)$ be the matrix of $(g-1)^{m+n-2k}$ with respect to the ordered K-bases $\mathcal{B}_{m+n-k}$ and $\mathcal{B}_k$ of D_{m+n-k} and D_k, respectively. Then*

$$A_k(m,n) = \begin{pmatrix} \binom{m+n-2k}{n-k} & \binom{m+n-2k}{n-k-1} & \cdots & \binom{m+n-2k}{n+1-2k} \\ \binom{m+n-2k}{n+1-k} & \binom{m+n-2k}{n-k} & \cdots & \binom{m+n-2k}{m+2-2k} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m+n-2k}{n-1} & \binom{m+n-2k}{n-2} & \cdots & \binom{m+n-2k}{n-k} \end{pmatrix}.$$

Proof. A typical element in $\mathcal{B}_{m+n-k}$ is $f_{m-k+t,n-t+1}$, $1 \le t \le k$, while a typical element in $\mathcal{B}_k$ is $f_{s,k+1-s}$, $1 \le s \le k$.

Now

$$(g-1)^{m+n-2k}(f_{m-k+t,n-t+1}) = \sum_{\ell=0}^{m+n-2k}\binom{m+n-2k}{\ell} f_{\ell+k+t-n,n-t+1-\ell}$$

When $\ell+k+t-n = s$ (and $n-t-\ell+1 = k+1-s$), $\ell = n+s-k-t$. Thus the coefficient $f_{s,k+1-s}$ in the expansion of $(g-1)^{m+n-2k}(f_{m-k+t,n-t+1})$ is

$$\binom{m+n-2k}{n+s-k-t}.$$

This proves our lemma. □

This matrix $A_k(m,n)$ has appeared in several papers, for example, [11, p. 145] and [6, Proof of Theorem 2].

3. Statement of Result

For an integer $k \in [1,m]$, define the $k\times 1$ column vector C_k to be the coordinate matrix of of x_k with respect to the basis $\mathcal{B}_k$ of D_k. Then C_k consists of alternating 1's and -1's. Then define the $k\times 1$ column vector B_k by $B_k = \operatorname{adj}(A_k(m,n))C_k$, where $\operatorname{adj}(A_k(m,n))$ is the classical adjoint of $A_k(m,n)$ (so $A_k(m,n)\operatorname{adj}(A_k(m,n)) = (\det A_k(m,n))I_k = \operatorname{adj}(A_k(m,n))A_k(m,n)$).

Theorem 1. *With $A_k(m,n)$ and B_k defined as above, and y_k defined by*

$$y_k = \sum_{i=1}^{k} b_{i1} f_{n-k+i,m+1-i},$$

the equation $(g-1)^{n+m-2k}(y_k) = (\det A_k(m,n))x_k$ holds. In addition, when $\lambda(m,n,p)$ is standard, $A_k(m,n)$ is invertible for every integer $k \in [1,m]$, $KGy_k \cong V_{n+m+1-2k}$ $(1 \le k \le m)$ and

$$V_m \otimes V_n = KGy_1 \oplus KGy_2 \oplus \cdots \oplus KGy_m.$$

4. Example

We illustrate Theorem 1 when $m = 4$, and $n = 5$.

When $k = 1$, $x_1 = f_{1,1}$, $A_1(4,5) = (\binom{7}{4}) = (35)$, $\det A_1(4,5) = 35$, and $\operatorname{adj}(A_1(4,5))C_1 = (1)(1) = (1)$. Thus $y_1 = f_{4,5}$, and

$$(g-1)^7(y_1) = \sum_{\ell=0}^{7} \binom{7}{\ell} f_{\ell-3,5-\ell} = \binom{7}{4} f_{1,1} = (\det A_1(4,5))x_1.$$

.

When $k = 2$, $x_2 = f_{1,2} - f_{2,1}$,

$$A_2(4,5) = \begin{pmatrix} \binom{5}{3} & \binom{5}{2} \\ \binom{5}{4} & \binom{5}{3} \end{pmatrix} = \begin{pmatrix} 10 & 10 \\ 5 & 10 \end{pmatrix},$$

$\det A_2(4,5) = 50$, and

$$\operatorname{adj}(A_2(4,5))C_2 = \begin{pmatrix} 10 & -10 \\ -5 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -15 \end{pmatrix}.$$

Thus $f_2 = 20f_{3,5} - 15f_{4,4}$, and

$$\begin{aligned}(g-1)^5(y_2) &= 20\sum_{\ell=0}^{5}\binom{5}{\ell}f_{\ell-2,5-\ell} - 15\left(\sum_{\ell=0}^{5}\binom{5}{\ell}f_{\ell-1,4-\ell}\right)\\ &= 20\left(\binom{5}{3}f_{1,2} + \binom{5}{4}f_{2,1}\right) - 15\left(\binom{5}{2}f_{1,2} + \binom{5}{3}f_{2,1}\right)\\ &= 50f_{1,2} - 50f_{2,1}\\ &= (\det A_2(4,5))x_2.\end{aligned}$$

When $k = 3$, $x_3 = f_{1,3} - f_{2,2} + f_{3,1}$,

$$A_3(4,5) = \begin{pmatrix}\binom{3}{2} & \binom{3}{1} & \binom{3}{0}\\ \binom{3}{3} & \binom{3}{2} & \binom{3}{1}\\ \binom{3}{4} & \binom{3}{3} & \binom{3}{2}\end{pmatrix} = \begin{pmatrix}3 & 3 & 1\\ 1 & 3 & 3\\ 0 & 1 & 3\end{pmatrix},$$

$\det A_3(4,5) = 10$, and

$$\operatorname{adj}(A_3(4,5))C_3 = \begin{pmatrix}6 & -8 & 6\\ -3 & 9 & -8\\ 1 & -3 & 6\end{pmatrix}\begin{pmatrix}1\\ -1\\ 1\end{pmatrix} = \begin{pmatrix}20\\ -20\\ 10\end{pmatrix}.$$

Thus $y_3 = 20f_{2,5} - 20f_{3,4} + 10f_{4,3}$ and

$$\begin{aligned}&(g-1)^3\cdot(20f_{2,5} - 20f_{3,4} + 10f_{4,3})\\ &\quad= 20\sum_{k=0}^{3}\binom{3}{k}f_{k-1,5-k} - 20\sum_{k=0}^{3}\binom{3}{k}f_{k,4-k} + 10\sum_{k=0}^{3}\binom{3}{k}f_{k+1,3-k}\\ &\quad= 20(3f_{1,3} + f_{2,2}) - 20(3f_{1,3} + 3f_{2,2} + f_{3,1}) = 10(f_{1,3} + 3f_{2,2} + 3f_{3,1})\\ &\quad= 10f_{1,3} - 10f_{2,2} + 10f_{3,1}\\ &\quad= (\det A_3(4,5))x_3.\end{aligned}$$

When $k = 4$, $x_4 = f_{1,4} - f_{2,3} + f_{3,2} - f_{4,1}$,

$$A_4(4,5) = \begin{pmatrix}\binom{1}{1} & \binom{1}{0} & \binom{1}{-1} & \binom{1}{2}\\ \binom{1}{2} & \binom{1}{1} & \binom{1}{0} & \binom{1}{-1}\\ \binom{1}{3} & \binom{1}{2} & \binom{1}{1} & \binom{1}{0}\\ \binom{1}{2} & \binom{1}{3} & \binom{1}{2} & \binom{1}{1}\end{pmatrix} = \begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix},$$

$\det A_4(4,5) = 1$, and

$$\operatorname{adj}(A_4(4,5))C_4 = \begin{pmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -1 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\\ 1\\ -1\end{pmatrix} = \begin{pmatrix}4\\ -3\\ 2\\ -1\end{pmatrix}.$$

Thus $y_4 = 4f_{1,5} - 3f_{2,4} + 2f_{3,3} - f_{4,2}$ and

$$\begin{aligned}(g-1)(y_4) &= 4f_{1,4} - 3(f_{1,4} + f_{2,3}) + 2(f_{2,3} + f_{3,2}) - (f_{3,2} + f_{4,1})\\ &= f_{1,4} - f_{2,3} + f_{3,2} - f_{4,1}\\ &= (\det A_4(4,5))x_4.\end{aligned}$$

Now $\lambda(4,5,p)$ is standard iff $p \neq 2,5,7$ by [5] and

$$V_4 \otimes V_5 = KGy_1 \oplus KGy_2 \oplus KGy_3 \oplus KGy_4$$

when $\lambda(4,5,p)$ is standard by Theorem 1.

5. Proof of Theorem 1

The author's original proof that $A_k(m,n)$ is invertible when $\lambda(m,n,p)$ is standard was far too long. The proof of this part of Theorem 1 given below was pointed out by an anonymous referee to whom the author is extremely grateful.

Proof of Theorem 1. Since by [2, p. 392],

$$[(g-1)^{n+m-2k} \cdot y_k]_{\mathcal{B}_k} = [(g-1)^{n+m-2k}]_{\mathcal{B}_k,\mathcal{B}_{m+n-k}}[y_k]_{\mathcal{B}_{m+n-k}},$$

we have

$$[(g-1)^{n+m-2k} \cdot y_k]_{\mathcal{B}_k} = A_k(m,n)B_k = A_k(m,n)\mathrm{adj}(A_k(m,n))C_k = (\det A_k(m,n))C_k.$$

But $[x_k]_{\mathcal{B}_k} = C_k$, which implies that $(g-1)^{n+m-2k}(y_k) = (\det A_k(m,n))x_k$.

We now follow the exposition of Iima and Iwamatsu's algorithm for computing the parts of $\lambda(m,n,p)$ in [7, Section 4]. Denote $\det A_k(m,n)$ by $D_k(m,n)$ when $1 \leq k \leq m$, set $D_0(m,n) = 1$, and note that $D_m(m,n) = 1$ because $A_m(m,n)$ is unipotent upper-triangular. Suppose that all the values of k with $1 \leq k \leq m$ satisfying $\delta_k(m,n,p) = 1$ are $0 = k_0 < k_1 < \cdots < k_t = m$. Then, by [7, Proposition 11],

$$V_m \oplus V_n \cong \bigoplus_{i=1}^{t} (k_i - k_{i-1})V_{m+n-k_i-k_{i-1}}.$$

Now assume that $\lambda(m,n,p)$ is standard. Then $t = m$, and $k_i = i$ for every integer $i \in [1,m]$. Thus, from the definition of the k_i, $\delta_{k_i} = \delta_i$ is non-zero in the field K for every $i \in [1,m]$. Hence $A_k(m,n)$ is invertible for every integer $k \in [1,m]$.

Since $\det A_k(m,n) \neq 0$, $(g-1)^{m+m-2k}(y_k) = (\det A_k(m,n))x_k$, and $(g-1)(x_k) = 0$, the indecomposable KG-submodule KGy_k has dimension $n+m-2k+1$ and one-dimensional socle $KGx_k = Kx_k$. Since the socles are linearly independent over K by Lemma 2, it follows that $KGy_1 + KGy_2 + \cdots + KGy_m$ is actually a direct sum. Finally, because the dimension of $KGy_1 + KGy_2 + \cdots + KGy_m$ over K is $\sum_{k=1}^{m}(n+m-2k+1) = nm$,

$$V_m \otimes V_n = KGy_1 + KGy_2 + \cdots + KGy_m.$$

□

References

[1] J. L. Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics 11, Cambridge University Press, Cambridge, 1986.
[2] H. Anton, *Elementary Linear Algebra*, 8th edition (2000), John Wiley & Sons, Inc.
[3] M. J. J. Barry, *Decomposing Tensor Products ans Exterior and Symmetric Squares*, Journal of Group Theory **14** (2011), 59–82.
[4] M. J. J. Barry, *Generators for Decompositions of Tensor Modules*, Archiv der Mathematik **97** (2011), no. 6, 503–512.

[5] M. J. J. Barry, *On a Question of Glasby, Praeger, and Xia*, Communications in Algebra **43** (2015), 4231–4246.
[6] S. P. Glasby, C.E. Praeger, and Binzhou Xia, *Decomposing modular tensor products, and periodicity of 'Jordan partitions'*, submitted.
[7] S. P. Glasby, C.E. Praeger, and Binzhou Xia, *Decomposing modular tensor products: 'Jordan partitions',their parts and p-parts*, accepted Israel J. Math.
[8] A. Granville, *Arithmetic Properties of Binomial Coefficients I: Binomial coefficients modulo prime powers*, Canadian Mathematical Society Conference Proceedings **20** (1997), 253–275.
[9] J. A. Green, *The modular representation algebra of a finite group*, Illinois Journal of Mathematics **6** (1962), 607–619.
[10] Xiang-Dong Hou, *Elementary divisors of tensor products and p-ranks of binomial matrices*, Linear Algebra and Its Applications **374** (2003), 255–274.
[11] K-i. Iima and R. Iwamatsu, *On the Jordan decomposition of tensored matrices of Jordan canonical forms*, Math. J. Okayama Univ. **51** (2009), 133–148.
[12] J. H. Lindsey, II, *Groups With a T. I. Cyclic Subgroup*, Journal of Algebra **30** (1974), 181–235.
[13] C. W. Norman, *On the Jordan form of the tensor product over fields of prime characteristic*, Linear and Multilinear Algebra **38** (1995), 351–371.
[14] C. W. Norman, *On Jordan bases for the tensor product and Kronecker sum and their elementary divisors over fields of prime characteristic*, Linear and Multilinear Algebra **56** (2008), 415–451.
[15] J.-C. Renaud, *The decomposition of products in the modular representation ring of a cyclic group of prime power order*, Journal of Algebra **58** (1979), 1–11.
[16] B. Srinivasan, *The modular representation ring of a cyclic p-group*, Proceedings of the London Mathematical Society (3) **14** (1964), 677-688.
[17] A. Trampus, *A canonical basis for the matrix transformation* $X \to AXB$, J. Math. Anal. Appl. **14** (1966), 153–160.
[18] A. Trampus, *A canonical basis for the matrix transformation* $X \to AX + XB$, J. Math. Anal. Appl. **14** (1966), 242–252.

Department of Mathematics, Allegheny College, Meadville, PA 16335

E-mail address: mbarry@allegheny.edu